1 Mathematical Diagrams

1.1 Mileage Charts

In this section we look at mileage charts.

Example 1

Distances in the table below are given in miles.

<table>
<thead>
<tr>
<th></th>
<th>BARNSTAPLE</th>
<th>BRISTOL</th>
<th>EXETER</th>
<th>PENZANCE</th>
<th>PLYMOUTH</th>
<th>TAUNTON</th>
</tr>
</thead>
<tbody>
<tr>
<td>BARNSTAPLE</td>
<td>100</td>
<td>55</td>
<td>108</td>
<td>67</td>
<td>194</td>
<td>125</td>
</tr>
<tr>
<td>BRISTOL</td>
<td>55</td>
<td>84</td>
<td>194</td>
<td>110</td>
<td>44</td>
<td>77</td>
</tr>
<tr>
<td>EXETER</td>
<td>108</td>
<td>194</td>
<td>44</td>
<td>77</td>
<td>34</td>
<td>144</td>
</tr>
<tr>
<td>PENZANCE</td>
<td>67</td>
<td>125</td>
<td>44</td>
<td>77</td>
<td>34</td>
<td>75</td>
</tr>
<tr>
<td>PLYMOUTH</td>
<td>50</td>
<td>51</td>
<td>144</td>
<td>75</td>
<td></td>
<td></td>
</tr>
<tr>
<td>TAUNTON</td>
<td>34</td>
<td>125</td>
<td>34</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Using the table, answer the following questions:

(a) How far is it from Taunton to Exeter?

(b) Jerry travels from Barnstaple to Exeter, then from Exeter to Plymouth, and finally from Plymouth back to Barnstaple.

How far does he travel altogether?

Solution

(a) 34 miles (see table and diagram opposite).

(b) Barnstaple to Exeter: 55 miles

Exeter to Plymouth: 44 miles

Plymouth to Barnstaple: 67 miles

Total distance = 55 + 44 + 67

= 166 miles
Example 2

The network diagram opposite shows the distances, in miles, between some towns.

Copy and complete the following mileage chart to show the shortest distances between these towns:

Solution

The direct distances can be completed first:

the shortest route from Amesbury to Devizes is via Upavon, a total of 19 miles;

the shortest route from Devizes to Salisbury is via Shrewton, a total of 25 miles;

the shortest route from Salisbury to Upavon is via Amesbury, a total of 17 miles;

the shortest route from Shrewton to Upavon is via Amesbury, a total of 15 miles.

With this information, the table can now be completed, as shown opposite.
Exercises

1. Use the table opposite, where the distances are given in miles, to find out how far it is from:
   (a) Leeds to Lincoln,
   (b) Hull to York,
   (c) Leeds to Manchester,
   (d) Sheffield to Leeds,
   (e) Manchester to York.

2. Ross travels from Leeds to Manchester, then from Manchester to Sheffield and finally from Sheffield back to Leeds. Use the table in question 1 to calculate the total distance he travels.

3. Hannah drives from Bristol to Exeter, continues to Plymouth, on to Barnstaple and from there back to Bristol. Use the table in Example 1 to calculate the total distance she drives.

4. The table opposite gives the distances in kilometres between some towns in northern France.
   What is the distance between:
   (a) Alençon and Paris,
   (b) Reims and Orleans,
   (c) Rouen and Calais,
   (d) Paris and Reims,
   (e) Le Mans and Rouen?

5. Debbie drives from Calais to Paris and back while she is on holiday. Use the table in question 4 to calculate how far she travels altogether on this journey.

6. Laura travels from Calais to Paris, on to Alençon and then to Rouen before returning to Paris. Use the table in question 4 to calculate how far she travels altogether.
7. The diagram below shows the distances, in miles, between some junctions on the M2 motorway:

Copy and complete the chart below to show the shortest distances between junctions:

8. The following network diagram shows the distances, in miles, between some towns in Wales:

Use the information in the diagram to complete a copy of the table on the next page, giving the shortest distances between the towns.
9. The network diagram below shows the distances, in miles, by road between some towns close to the Scottish border:

Use information from the diagram to complete a copy of the table opposite, giving the shortest distances between the towns.
10. The diagram below shows stations on the GNER railway:

```
LONDON  STEVENAGE  PETERBOROUGH  DONCASTER  YORK
```

Some distances, in miles, are shown in the table opposite.

(a) Copy the table and fill in the missing distances.
(b) What distance is travelled in a return journey between London and York?

1.2 Using Flow Charts to Plan Practical Tasks

A flow chart can be used to organise the instructions for carrying out a task. Boxes of different shapes are used for particular operations:

- **START** or **END POINTS**
- **INSTRUCTION BOXES**
- **DECISION BOXES**

Each box contains only one instruction.
Example 1

Draw a flow chart to give the instructions for making a mug of tea.

**Solution**

The flow chart gives the instructions for making a mug of tea without sugar.

A mug of tea *with* sugar, needs this extra instruction box:

Note that the instruction box **PUT TEA BAG IN MUG** can go earlier in the flow chart, for example, immediately after the **START** box. The extra box **ADD SUGAR** can also go in various positions: for example, before or after the **PUT TEA BAG IN MUG** box, or before or after the **ADD MILK** box.

Example 2

The instruction **BOIL KETTLE** in Example 1 can be broken down into separate stages.

Draw a flow chart to show this.
Example 3

Draw a flow chart showing how to find a programme you would like to watch on television.

Solution

- Plug in kettle
- Switch on kettle
- Wait 20 seconds
- Is the kettle boiling?
- Yes
  - Watch the programme
- No
  - Change channels
- Have you tried all the channels?
- Yes
  - Watch the programme
- No
  - Change channels
- Is it a good programme?
- Yes
  - Watch the programme
- No
  - Have you tried all the channels?
  - Change channels
- Switch off and find something else to do

STOP
Exercises

1. Draw a flow chart showing how to prepare a drink of blackcurrant squash in a glass.

2. Draw a flow chart for each of the following:
   (a) making a cup of coffee with milk and sugar,
   (b) buying a can of drink from a vending machine,
   (c) making a telephone call from a pay phone,
   (d) shutting down a computer.

3. Draw a flow chart that describes how to cross a road. You should include decision boxes in your flow chart.

4. Imagine you are driving along a road. You see a 30 mph speed limit sign and a speed camera. Draw a flow chart that you, as a sensible driver, would be advised to follow.

5. Jerry needs to work out $4.72 \times 11.61$ using a calculator.
   (a) Draw a flow chart to show how to carry out this calculation on a calculator.
   (b) Redraw the flow chart to include all the following processes:
       (i) estimating the answer to $4.72 \times 11.61$,
       (ii) calculating the answer to $4.72 \times 11.61$
       (iii) comparing the answer with the estimate to decide whether the calculator answer is reasonable.

6. You are playing Snakes and Ladders.
   (a) Draw a flow chart to describe how to move your counter for one go.
   (b) Describe how you would change your chart when you have an extra turn after throwing a six.
1.3 Using a Flow Chart for Classification

Flow charts can also be used to sort or classify things.

Example 1

This flow chart can be used to classify angles between 0 ° and 360 °. What comes out at the end points A, B, C, D and E?

Solution

A  Angles less than 90 °  ACUTE ANGLES

B  Angles equal to 90 °  RIGHT ANGLES

C  Angles greater than 180 °  REFLEX ANGLES

D  Angles equal to 180 °  ANGLES ON STRAIGHT LINES

E  Angles greater than 90 ° but less than 180 °  OBTUSE ANGLES
Example 2

The flow chart below can be used for sorting quadrilaterals:

![Flow chart image]

Where would each of these shapes come out?

(a)  
![Shape image]

(b)  
![Shape image]

(c)  
![Shape image]

(d)  
![Shape image]

(e)  
![Shape image]

(f)  
![Shape image]
Solution
(a) A parallelogram; comes out at D.
(b) A square; comes out at A.
(c) A trapezium; comes out at E.
(d) Quadrilateral with no special properties; comes out at F.
(e) A rectangle; comes out at B.
(f) A rhombus; comes out at C.

Exercises
1. The angles below are classified using the flow chart in Example 1. Where does each angle come out of the flow chart?
(a) (b) (c)
(d) (e) (f)
(g)
2. The quadrilaterals below can be classified using the flow chart in Example 2. Where does each quadrilateral come out of the flow chart?

(a) (b) (c) (d) (e) (f)

3. Draw a flow chart that will sort trees, into those that shed their leaves during winter, and those that do not.

4. Draw a flow chart that will sort triangles into equilateral (all sides of equal length), isosceles (two sides of equal length) or scalene (all sides of different lengths).

5. Draw a flow chart that will sort polygons into the following categories:

   Triangles
   Quadrilaterals
   Pentagons
   Hexagons
   Heptagons
   Octagons
   Polygons with more than 8 sides

6. The flow chart on the following page can be used to sort animals.

   (a) Where would each of these animals come out of the flow chart?

   Monkey Bird Giraffe
   Horse Centipede Elephant
   Zebra Pig Kangaroo
   Fish Dolphin Cat
(b) Name one more animal that would come out of the flow chart at each of the end points A, B, C, D and E.

7. The following flow chart will sort numbers. The numbers 1 to 10 are put into this flowchart. Where do they each come out?
8. Draw a flow chart that will classify numbers as *odd* or *even*.

9. Draw a flow chart that will test numbers up to 20 to see if they are prime.

### 1.4 Networks

In this section we consider networks. Problems can be solved by finding the shortest or quickest route through a network, which can represent a system of roads, pipelines, cables, or anything else that connects different points or places.

**Example 1**

Find the shortest route from A to G, using the distances shown on the network opposite:

![Network Diagram](image)

**Solution**

First find the shortest distances from A to the points B, C and D; these are shown in circles on this diagram:

![Distance Diagram](image)

Now work out the shortest distances from A to E and F and put these in circles on the diagram:

![Distance Diagram](image)

It is now possible to see that the shortest route from A to G is 10, using the route, A B E G.
Example 2

Find the shortest route from S to T through this network:

Solution

The diagram opposite shows the shortest routes from S to A, C and D:

The shortest routes from S to B and E are now added to the diagram:

We can see that the shortest route from S to T is 6, using either route S D C E T or route S A C E T.
Exercises

1. The network diagram below shows the distances between some towns and cities:

![Network Diagram]

Find the shortest distance and route between:

(a) Manchester and Nottingham,
(b) Sheffield and Birmingham.

2. The diagram below shows the shortest journey times, in hours, between 5 towns, A, B, C, S and T:

![Journey Times Diagram]

Find the shortest journey time from S to T and state the route.

3. The network diagram below shows the distances between some towns and cities in the south west of England:

![Network Diagram]
Find the shortest distance and route between:
(a) Bodmin and Exeter,
(b) Plymouth and Minehead.

4. The network diagram below shows some of the places on the Isle of Man, and the distances between them:

Which is the shortest route and what is the distance from:
(a) Douglas to Sulby,
(b) Ramsey to Port Erin,
(c) Peel to Laxey?

5. Use the network diagram below to find both the shortest route and the distance from Carlisle to Lincoln.
1.5 Critical Path Analysis

When you are planning to carry out a task, critical path analysis can be used to help you find the most efficient way to do it; this works by showing how activities need to be scheduled.

For example, when making your breakfast, you can boil the kettle and cook your toast at the same time. You do not have to wait until you have boiled the kettle before you start to make your toast, whereas you do have to boil the kettle before you can make a cup of tea.

Example 1

Veronica is going to make a cake. She has six tasks to do, which are listed below:

<table>
<thead>
<tr>
<th>Activity</th>
<th>Time needed in minutes</th>
<th>Preceded by</th>
</tr>
</thead>
<tbody>
<tr>
<td>A Warm oven</td>
<td>15</td>
<td></td>
</tr>
<tr>
<td>B Weigh ingredients</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>C Mix ingredients</td>
<td>5</td>
<td>Weigh ingredients</td>
</tr>
<tr>
<td>D Bake cake</td>
<td>20</td>
<td>Mix ingredients</td>
</tr>
<tr>
<td>E Wash up mixing bowl and</td>
<td>8</td>
<td>Mix ingredients</td>
</tr>
<tr>
<td>utensils</td>
<td></td>
<td></td>
</tr>
<tr>
<td>F Wash up cake tin</td>
<td>2</td>
<td>Bake cake</td>
</tr>
</tbody>
</table>

Draw an activity network and find the shortest time to make the cake.

Solution

The first step is to draw an activity network, which is a way of showing the data concerning the tasks that have to be completed, how long each one takes, and the order in which they must be undertaken. The activity network for making Veronica's cake is shown below:
We move through the network from *left to right*, calculating the earliest start time for each activity. For example, D (bake cake) cannot begin until A, B and C have all happened, i.e. the oven is warm and the cake ingredients have been weighed out and mixed. The oven requires at least 15 minutes to warm, so the earliest time that D and E can start is 15 minutes after warming the oven. In turn, this means that F cannot start until 35 minutes have elapsed.

At this stage, we can see that the shortest time to complete the task of baking the cake is 37 minutes.

We then work through the network from *right to left*, calculating the latest start time for each activity. For example, E could begin 29 minutes after the start, and still finish at the same time as F. However, D must begin no later than 15 minutes after the start, for the whole activity to be completed in 37 minutes.

The *critical path* is A D F as these tasks must be completed on time or the whole project will be delayed.

For activity E there is a *float time* of 14 minutes. We use this term because there is a 22 minute period in which the activity must take place, but the task itself takes only 8 minutes. C also has a float time, but this time of 7 minutes duration.

There is *no float time* for tasks on the critical path.
Example 2

On the following activity network the numbers show the time, in minutes, for each task. Complete the earliest and latest finishing times for each task, identify the critical path, and state the shortest completion time for the whole activity.

Solution

The layout of the activity network, from left to right, tells us the following information about the order in which the tasks must be carried out:

<table>
<thead>
<tr>
<th>Task</th>
<th>Time (minutes)</th>
<th>Preceded by</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>B</td>
<td>3</td>
<td>A</td>
</tr>
<tr>
<td>C</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>D</td>
<td>4</td>
<td>C</td>
</tr>
<tr>
<td>E</td>
<td>2</td>
<td>A, B</td>
</tr>
<tr>
<td>F</td>
<td>2</td>
<td>A, B</td>
</tr>
<tr>
<td>G</td>
<td>8</td>
<td>A, B</td>
</tr>
<tr>
<td>H</td>
<td>8</td>
<td>A, B, C, D, E</td>
</tr>
<tr>
<td>I</td>
<td>7</td>
<td>A, B, F</td>
</tr>
</tbody>
</table>
We first calculate the earliest start time for each activity by working from *left to right* through the activity network:

We now work back from *right to left*, calculating the latest start time for each activity:

So the critical path is \( C \rightarrow D \rightarrow H \), because for this route the earliest start times match the latest start times.

The shortest completion time is 16 minutes.

The diagram shows that there are float times for some activities; for example, 2 minutes for activity I, and 1 minute for both activities F and B.
Example 3

The table lists the tasks needed to completely refurbish a kitchen; the times are given in days. Find both the critical path, and the shortest completion time.

<table>
<thead>
<tr>
<th>Task</th>
<th>Time needed</th>
<th>Preceded by</th>
</tr>
</thead>
<tbody>
<tr>
<td>A Design kitchen</td>
<td>8</td>
<td></td>
</tr>
<tr>
<td>B Make kitchen units</td>
<td>11</td>
<td>A</td>
</tr>
<tr>
<td>C Remove old units</td>
<td>2</td>
<td>A</td>
</tr>
<tr>
<td>D Fit new power points</td>
<td>1</td>
<td>C</td>
</tr>
<tr>
<td>E Fit new plumbing</td>
<td>2</td>
<td>C</td>
</tr>
<tr>
<td>F Paint and decorate</td>
<td>3</td>
<td>D, E</td>
</tr>
<tr>
<td>G Fit new units</td>
<td>5</td>
<td>B, F</td>
</tr>
<tr>
<td>H Fix wall tiles</td>
<td>3</td>
<td>G</td>
</tr>
</tbody>
</table>

Solution

The activity network is shown below:

Now move from left to right through the network to find the earliest start times:
Now work back through the network, putting in the latest start times:

```
A: 0
B: 8
C: 10
D: 15
E: 16
F: 20
G: 21
H: 27
```

The critical path is A B G H, as shown in the diagram, and the shortest completion time is 27 days.

### Exercises

1. The activity diagram below shows the time, in minutes, for different parts of a process. Find the critical path and the shortest possible completion time.

   ![Activity Diagram]

2. Jamil and Halim are making a model garage for their brother's birthday present. There are a number of tasks that must be completed to make the garage; these are listed in the table opposite:

<table>
<thead>
<tr>
<th>Task</th>
<th>Time needed (hours)</th>
</tr>
</thead>
<tbody>
<tr>
<td>A Design the garage</td>
<td>1</td>
</tr>
<tr>
<td>B Buy materials</td>
<td>2</td>
</tr>
<tr>
<td>C Cut out wooden panels</td>
<td>2</td>
</tr>
<tr>
<td>D Glue panels in place</td>
<td>1</td>
</tr>
<tr>
<td>E Paint garage</td>
<td>2</td>
</tr>
<tr>
<td>F Make cars</td>
<td>3</td>
</tr>
<tr>
<td>G Paint cars</td>
<td>3</td>
</tr>
</tbody>
</table>
The activity network is shown below:

![Diagram]

Find the critical path and the shortest possible completion time.

3. The following instructions must be carried out to put up a dome tent:

<table>
<thead>
<tr>
<th>Task</th>
<th>Time (mins)</th>
<th>Preceded by</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>Peg down inner tent</td>
<td>3</td>
</tr>
<tr>
<td>B</td>
<td>Assemble poles</td>
<td>1</td>
</tr>
<tr>
<td>C</td>
<td>Fit poles in flysheet</td>
<td>4</td>
</tr>
<tr>
<td>D</td>
<td>Erect flysheet</td>
<td>4 A, B, C</td>
</tr>
<tr>
<td>E</td>
<td>Peg down flysheet</td>
<td>2 A, B, C, D</td>
</tr>
<tr>
<td>F</td>
<td>Hang inner tent from poles</td>
<td>2 A, B, C, D</td>
</tr>
<tr>
<td>G</td>
<td>Attach guy ropes</td>
<td>4 A, B, C, D, E</td>
</tr>
</tbody>
</table>

The activity network is shown below:

![Diagram]

Find both the critical path, and the shortest time needed to put up the tent.
4. You are going to prepare a meal of chicken and potato pie, peas and gravy. You have to carry out the tasks listed below:

<table>
<thead>
<tr>
<th>Task</th>
<th>Time (mins)</th>
<th>Preceded by</th>
</tr>
</thead>
<tbody>
<tr>
<td>A Peel potatoes</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>B Heat oven</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>C Make pie</td>
<td>12</td>
<td>A</td>
</tr>
<tr>
<td>D Cook pie</td>
<td>40</td>
<td>B, C</td>
</tr>
<tr>
<td>E Make gravy</td>
<td>5</td>
<td>B, C</td>
</tr>
<tr>
<td>F Cook peas</td>
<td>8</td>
<td>B, C</td>
</tr>
<tr>
<td>G Cook potatoes</td>
<td>25</td>
<td>A</td>
</tr>
<tr>
<td>H Lay table</td>
<td>2</td>
<td></td>
</tr>
</tbody>
</table>

(a) Draw an activity network in line with the information given in the table.
(b) Find both the critical path and the shortest time needed to prepare the meal.
(c) Which tasks have a float time? State the float times for these activities.

5. The building of a house is broken down into the tasks listed in the table below. Draw a network diagram and use it to find the critical path and the shortest possible construction time.

<table>
<thead>
<tr>
<th>Task</th>
<th>Time (days)</th>
<th>Preceded by</th>
</tr>
</thead>
<tbody>
<tr>
<td>A Order materials</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>B Lay drains</td>
<td>7</td>
<td>A</td>
</tr>
<tr>
<td>C Lay foundations</td>
<td>7</td>
<td>B</td>
</tr>
<tr>
<td>D Erect blockwork</td>
<td>11</td>
<td>C</td>
</tr>
<tr>
<td>E Roofing work</td>
<td>5</td>
<td>D</td>
</tr>
<tr>
<td>F Install floors</td>
<td>4</td>
<td>E</td>
</tr>
<tr>
<td>G Plumbing and heating</td>
<td>10</td>
<td>F</td>
</tr>
<tr>
<td>H Electrical installation</td>
<td>6</td>
<td>F</td>
</tr>
<tr>
<td>I Install windows</td>
<td>5</td>
<td>D</td>
</tr>
<tr>
<td>J Plastering</td>
<td>6</td>
<td>G, H, I</td>
</tr>
<tr>
<td>K Decoration</td>
<td>5</td>
<td>J</td>
</tr>
<tr>
<td>L Install fittings</td>
<td>3</td>
<td>K</td>
</tr>
<tr>
<td>M Clear site</td>
<td>2</td>
<td>L</td>
</tr>
<tr>
<td>N Lay paths</td>
<td>2</td>
<td>D</td>
</tr>
</tbody>
</table>
2 Factors

2.1 Factors and Prime Numbers

A factor divides exactly into a number, leaving no remainder. For example, 13 is a factor of 26 because $26 \div 13 = 2$ leaving no remainder.

A prime number has only two factors, 1 and itself; this is how a prime number is defined.

5 is a prime number because it has only two factors, 1 and 5.

8 has factors 1, 2, 4 and 8, so it is not prime.

1 is not a prime number because it has only one factor, namely 1 itself.

Example 1

(a) List the factors of the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9 and 10.

(b) Which of these numbers are prime numbers?

Solution

(a) This table lists the factors of these numbers:

<table>
<thead>
<tr>
<th>Number</th>
<th>Factors</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>1, 2</td>
</tr>
<tr>
<td>3</td>
<td>1, 3</td>
</tr>
<tr>
<td>4</td>
<td>1, 2, 4</td>
</tr>
<tr>
<td>5</td>
<td>1, 5</td>
</tr>
<tr>
<td>6</td>
<td>1, 2, 3, 6</td>
</tr>
<tr>
<td>7</td>
<td>1, 7</td>
</tr>
<tr>
<td>8</td>
<td>1, 2, 4, 8</td>
</tr>
<tr>
<td>9</td>
<td>1, 3, 9</td>
</tr>
<tr>
<td>10</td>
<td>1, 2, 5, 10</td>
</tr>
</tbody>
</table>

(b) The numbers 2, 3, 5 and 7 have exactly two factors, and so only they are prime numbers.
Example 2
List the prime factors of 24.

**Solution**
First list all the factors of 24, and they are:

\[1, 2, 3, 4, 6, 8, 12, 24\]

Now select from this list the numbers that are prime; these are 2 and 3, and so the prime factors of 24 are 2 and 3.

Example 3
Which of the following numbers are prime numbers:

\[18, 45, 79 \text{ and } 90\]?

**Solution**
The factors of 18 are 1, 2, 3, 6, 9 and 18; 18 is not a prime number.
The factors of 45 are 1, 3, 5, 9, 15 and 45; 45 is not a prime number.
The factors of 79 are 1 and 79; 79 is a prime number.
The factors of 90 are 1, 2, 3, 5, 6, 9, 10, 15, 18, 30, 45 and 90; 90 is not a prime number.

79 is the only prime number in the list.

**Divisibility Test**

- If a number is divisible by 2, it will end with 0, 2, 4, 6 or 8.
- If a number is divisible by 3, the sum of its digits will be a multiple of 3.
- If a number is divisible by 4, the last two digits will be a multiple of 4.
- If a number is divisible by 5, it will end in 0 or 5.
- If a number is divisible by 9, the sum of its digits will be a multiple of 9.
- If a number is divisible by 10, it will end in 0.

*Can you find tests for divisibility by other numbers?*
Exercises

1. (a) List all the factors of each of the following numbers:
   11, 12, 13, 14, 15, 16, 17, 18, 19, 20
   (b) Which of these numbers are prime?

2. Explain why 99 is not a prime number.

3. Which of the following are prime numbers:
   33, 35, 37, 39?

4. Find the prime factors of 72.

5. (a) Find the prime factors of 40.
   (b) Find the prime factors of 70.
   (c) Which prime factors do 40 and 70 have in common?

6. Find the prime factors that 48 and 54 have in common.

7. A number has prime factors 2, 5 and 7. Which is the smallest number that has these prime factors?

8. The first 5 prime numbers are 2, 3, 5, 7 and 11. Which is the smallest number that has these prime factors?

9. Write down the first two prime numbers which are greater than 100.

10. Which is the first prime number that is greater than 200?
2.2 Prime Factors

A factor tree may be used to help find the prime factors of a number.

Example 1

Draw a factor tree for the number 36.

Solution

Start with 36 and then:

split 36 into numbers 9 and 4 that multiply to give 36 as shown in the factor tree opposite;

repeat for the 9 and the 4, as shown on the factor tree.

The factor tree is now complete because the numbers at the ends of the branches are prime numbers; the prime numbers have been ringed.

Another possible factor tree for 36 is shown here:

On the factor tree we only put a ring around the prime numbers.

Note that, at the end of the branches, both the numbers 2 and 3 appear twice.
The prime factors of 36 are 3, 2, 2 and 3.
In ascending order, the prime factors of 36 are 2, 2, 3, 3.

From the factor trees above it is possible to write:

\[ 36 = 2 \times 2 \times 3 \times 3 \]
\[ = 2^2 \times 3^2 \]

When a number is written in this way, it is said to be written as the product of its prime factors.
Example 2

Express each of the following numbers as the product of its prime factors:

(a) 102
(b) 60

Solution

(a) Start by creating a factor tree:

\[
\begin{align*}
102 & \rightarrow 2 \times 3 \times 17 \\
51 & \rightarrow 3 \times 17 \\
2 & \rightarrow 2.
\end{align*}
\]

(b) Start by creating a factor tree:

\[
\begin{align*}
60 & \rightarrow 5 \times 3 \times 2 \times 2 \\
15 & \rightarrow 5 \times 3 \\
4 & \rightarrow 2 \times 2.
\end{align*}
\]

Put the prime numbers in ascending order:

\[
\begin{align*}
60 & = 2 \times 2 \times 3 \times 5 \\
& = 2^2 \times 3 \times 5
\end{align*}
\]

Example 3

A number is expressed as the product of its prime factors as

\[2^3 \times 3^2 \times 5\]

What is the number?

Solution

\[
\begin{align*}
2^3 \times 3^2 \times 5 & = 2 \times 2 \times 2 \times 3 \times 3 \times 5 \\
& = 360
\end{align*}
\]
Exercises

1. Draw factor trees for the following numbers:
   (a) 20  (b) 100  (c) 88

2. Draw two different factor trees for 40.

3. (a) Draw two different factor trees for 66.
     (b) Can you draw any other different factor trees for 66?

4. Copy the factor tree opposite and fill in the missing numbers:

5. Fill in the missing numbers on a copy of the factor tree opposite:

6. Use a factor tree to find the prime factors of:
   (a) 30  (b) 80  (c) 200

7. Write each of the following numbers as the product of their prime factors:
   (a) 62  (b) 64  (c) 82
   (d) 320  (e) 90  (f) 120
   (g) 54  (h) 38  (i) 1000

8. A number is expressed as the product of its prime factors as:
   \[2^3 \times 3 \times 5^2\]
   What is the number?
9. The prime factors of a number are 2, 7 and 11. Which are the three smallest numbers with these prime factors?

10. Which is the smallest number that has:
   (a) 4 different prime factors,
   (b) 5 prime factors?

11. (a) Write down two numbers, neither of which must end in 0, and which multiply together to give 1000.
   (b) Repeat question 11 (a), this time writing down two numbers which multiply to give 1 000 000.

2.3 Index Notation

You will have seen the occasional use of index notation in the last section; for example, in the statement

\[ 2^3 \times 3^2 \times 5 = 360 \]

which contains 2 indices.

We read \(2^3\) as "two to the power of three" or "two cubed": 2 is the base number; 3 is the index.

In general, \(a^n\) is the result of multiplying the base number, \(a\), by itself \(n\) times, \(n\) being the index.

\[ a^n = a \times a \times a \times \ldots \times a \times a \times a \quad n \text{ times} \]

A calculator can be used to work out powers. The index button is usually marked \(x^y\) or \(y^x\). Sometimes you will need to press the SHIFT or 2nd FUNCTION key before using the index button. You should find out which buttons you need to use on your calculator.

For example, to calculate \(5^4\) you may need to press

either \(5 \times^y 4 =\)

or \(5 \text{ SHIFT} x^y 4 =\)

to get the correct answer of 625.
Example 1

Calculate:
(a) \(2^4\)  
(b) \(7^3\)  
(c) \(10^5\)

Check your answers using a calculator.

Solution

(a) \(2^4 = 2 \times 2 \times 2 \times 2 = 16\)

Using a calculator,
either \(2 \text{ SHIFT } 4 = 16\)
or \(2 \times^4 4 = 16\)

(b) \(7^3 = 7 \times 7 \times 7 = 343\)

Using a calculator,
either \(7 \text{ SHIFT } 3 = 343\)
or \(7 \times^3 3 = 343\)

(c) \(10^5 = 10 \times 10 \times 10 \times 10 \times 10 = 100000\)

Using a calculator,
either \(1 \text{ 0 SHIFT } 5 = 100000\)
or \(1 \times^5 5 = 100000\)

Example 2

Write these statements, filling in the missing numbers:

(a) \(32 = 2^\square\)  
(b) \(1000000 = 10^\square\)

Solution

(a) \(32 = 2 \times 2 \times 2 \times 2 \times 2 = 2^5\)

(b) \(1000000 = 10 \times 10 \times 10 \times 10 \times 10 \times 10 = 10^6\)
Exercises

1. Copy the following statements and fill in the missing numbers:
   (a) \[6 \times 6 \times 6 \times 6 \times 6 = 6\]  
   (b) \[3 \times 3 \times 3 \times 3 = 3\]  
   (c) \[7 \times 7 \times 7 \times 7 \times 7 = 7\]  
   (d) \[9 \times 9 \times 9 \times 9 \times 9 = 9\]

2. Calculate:
   (a) \[2^3\]  
   (b) \[3^3\]  
   (c) \[10^4\]  
   (d) \[5^3\]  
   (e) \[2^7\]  
   (f) \[3^4\]  
   (g) \[9^2\]  
   (h) \[10^3\]  
   (i) \[10^7\]

3. Copy the following statements and fill in the missing numbers:
   (a) \[100 = 10\]  
   (b) \[81 = 2\]  
   (c) \[81 = 3\]  
   (d) \[16 = 4\]  
   (e) \[16 = 2\]  
   (f) \[7 = 2401\]

4. Calculate:
   (a) \[5^2 \times 2^2\]  
   (b) \[3^2 \times 2^4\]  
   (c) \[7^2 \times 2^3\]  
   (d) \[6^2 \times 2\]  
   (e) \[9^2 \times 3\]  
   (f) \[5^3 \times 2^1\]

5. Copy each of the following statements and fill in the missing numbers:
   (a) \[2^3 \times 2^5 = (2 \times 2 \times 2) \times (2 \times 2 \times 2 \times 2 \times 2)\]  
   \[= 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2\]  
   \[= 2\]  
   (b) \[5^7 \times 5^2 = (5 \times 5 \times 5 \times 5 \times 5 \times 5 \times 5) \times (5 \times 5)\]  
   \[= 5 \times 5 \times 5 \times 5 \times 5 \times 5 \times 5 \times 5 \times 5 \times 5\]  
   \[= 5\]  
   (c) \[6^4 \times 6^2 = 6\]  
   (d) \[7^3 \times 7^7 = 7\]  
   (e) \[8^6 \times 8 = 8\]
6. Copy the following statements and fill in the missing numbers:
   (a) \[ 9^3 = 9 \times 9 \times 9 \]
       \[ = (3 \times 3) \times (3 \times 3) \times (3 \times 3) \]
       \[ = 3 \times 3 \times 3 \times 3 \times 3 \times 3 \]
       \[ = 3 \square \]
   (b) \[ 9^4 = 3 \square \]

7. If \( 4^2 = 2^n \), which number does \( n \) represent? Answer this question using a similar method to the one used in question 6.

8. If \( 4^n = 2^{12} \), which number does \( n \) represent?

9. If \( 125^4 = 5^n \), which number does \( n \) represent?

10. Copy the following statements and fill in the missing numbers:
    (a) \[ 5^4 \times 2^4 = 10 \square \]
    (b) \[ 3^5 \times 2^5 = 6 \square \]
    (c) \[ 4 \square \times 2^3 = 8^3 \]
    (d) \[ 7^5 \times 3^5 = 21 \square \]
    (e) \[ 7^4 \times \square^4 = 28^4 \]
    (f) \[ 5^9 \times \square^9 = 10^9 \]

2.4 Highest Common Factor and Lowest Common Multiple

The *highest common factor* (HCF) of two numbers is the largest number that is a factor of both.

The factors of 12 are 1, 2, 3, 4, 6, 12.
The factors of 15 are 1, 3, 5, 15.

So the HCF of 12 and 15 is 3.

The HCF is easy to find for some numbers, but for others it is more difficult. In harder cases, the best way to find the HCF is to use prime factors.
Example 1
Find the HCF of:

(a) 20 and 30  
(b) 14 and 12

Solution
(a) The factors of 20 are 1, 2, 4, 5, 10 and 20.
The factors of 30 are 1, 2, 3, 5, 6, 10, 15 and 30.
The HCF of 20 and 30 is 10.

(b) The factors of 14 are 1, 2, 7 and 14.
The factors of 12 are 1, 2, 3, 4, 6 and 12.
The HCF of 14 and 12 is 2.

Example 2
Find the HCF of 60 and 72.

Solution
Using factor trees:

\[ 60 = 2 \times 2 \times 3 \times 5 \]
\[ = 2^2 \times 3 \times 5 \]

\[ 72 = 2 \times 2 \times 2 \times 3 \times 3 \]
\[ = 2^3 \times 3^2 \]

The HCF is calculated using the prime factors that are common to both numbers. In this case, 2 appears twice in both, and 3 appears once in both.

So,
the HCF of 60 and 72 \[ = 2 \times 2 \times 3 \]
\[ = 12 \]
To be in the HCF, the prime factor must be in both lists:

\[
\begin{align*}
60 &= 2 \times 3 \times 5 \\
72 &= 2 \times 2 \times 3 \times 3 \\
\text{HCF} &= 2 \times 3 \\
\text{HCF} &= 12
\end{align*}
\]

Alternatively, using indices:

\[
\begin{align*}
60 &= 2^1 \times 3^1 \times 5^1 \\
72 &= 2^3 \times 3^2 \times 5^0 \\
\text{HCF} &= 2^1 \times 3^1 \\
\text{HCF} &= 12
\end{align*}
\]

The lowest common multiple (LCM) of two numbers is the smallest number that is a multiple of both.

For example, 18 is the smallest number that is a multiple of both 6 and 9, so the LCM of 6 and 9 is 18.

Example 3

What is the LCM of:

(a) 5 and 7 
(b) 6 and 10

Solution

(a) The multiples of 5 are:
5, 10, 15, 20, 25, 30, 35, 40, 45, ...

The multiples of 7 are:
7, 14, 21, 28, 35, 42, 49, ...

The LCM of 5 and 7 = 35.

(b) The multiples of 6 are:
6, 12, 18, 24, 30, 36, 42, ...

The multiples of 10 are:
10, 20, 30, 40, 50, 60, ...

The LCM of 6 and 10 = 30.
The LCM for larger numbers can be found by using prime factorisation.

**Example 4**
Find the LCM of 60 and 72.

**Solution**
From Example 2,

\[
60 = 2 \times 2 \times 3 \times 5 = 2^2 \times 3 \times 5 \quad \text{and} \quad 72 = 2 \times 2 \times 2 \times 3 \times 3 = 2^3 \times 3^2
\]

The LCM includes all the factors from either number.

To be in the LCM, the prime factor can be in either list or in both lists:

\[
\begin{align*}
60 &= 2 \times 2 \times 3 \times 5 \\
72 &= 2 \times 2 \times 2 \times 3 \times 3 \\
\text{LCM} &= 2 \times 2 \times 2 \times 3 \times 3 \times 5
\end{align*}
\]

LCM = 360

Alternatively, using indices:

\[
\begin{align*}
60 &= 2^2 \times 3 \times 5^1 \\
72 &= 2^3 \times 3^2 \\
\text{Highest power of 3} \\
\text{LCM} &= 2^3 \times 3^2 \times 5 \\
\text{Highest power of 2} \quad \text{Highest power of 5} \\
\text{LCM} &= 360
\end{align*}
\]

**Example 5**
Find the HCF and LCM of 50 and 70.

**Solution**
Using factor trees to find the prime factorisations:

\[
\begin{align*}
50 &= 2 \times 5 \times 5 \\
&= 2^1 \times 5^2 \\
70 &= 2 \times 5 \times 7 \\
&= 2^1 \times 5^1 \times 7^1
\end{align*}
\]
\[
\begin{align*}
\text{HCF} & = 2^1 \times 5^1 \times 7^0 \\
& = 10 \\
\text{LCM} & = 2^1 \times 5^2 \times 7^1 \\
& = 350
\end{align*}
\]

Exercises

1. (a) List the factors of 21.
   (b) List the factors of 35.
   (c) What is the HCF of 21 and 35?

2. Find the HCF of:
   (a) 6 and 9
   (b) 14 and 18
   (c) 30 and 24
   (d) 15 and 10

3. (a) Use a factor tree to find the prime factorisation of 42.
   (b) Use a factor tree to find the prime factorisation of 90.
   (c) Find the HCF of 42 and 90.

4. What is the HCF of:
   (a) 90 and 120
   (b) 96 and 72
   (c) 56 and 60
   (d) 77 and 50
   (e) 300 and 550
   (f) 320 and 128?

5. (a) List the first 10 multiples of 8.
   (b) List the first 10 multiples of 6.
   (c) What is the LCM of 6 and 8?

6. What is the LCM of:
   (a) 5 and 3
   (b) 9 and 6
   (c) 8 and 10
   (d) 12 and 9
   (e) 15 and 20
   (f) 6 and 11?

7. (a) Use a factor tree to find the prime factorisation of 66.
   (b) Use a factor tree to find the prime factorisation of 40.
   (c) Find the LCM of 40 and 66.
8. Find the LCM of:
   (a) 28 and 30      (b) 16 and 24      (c) 20 and 25
   (d) 60 and 50      (e) 12 and 18      (f) 21 and 35

9. Two lighthouses can be seen from the top of a hill. The first flashes once every 8 seconds, and the other flashes once every 15 seconds. If they flash simultaneously, how long is it until they flash again at the same time?

10. At a go-kart race track, Vic completes a lap in 40 seconds; Paul completes a lap in 30 seconds, and Mark completes a lap in 50 seconds.
    If all three start a lap at the same time, how long is it before
    (a) Paul overtakes Vic,
    (b) Vic overtakes Mark?

2.5 Squares and Square Roots

To square a number you multiply the number by itself.

If you square 8, you multiply 8 by 8:

\[ 8 \times 8 = 64 \]

so the square of 8 = 64.

The calculator button for squaring numbers usually looks like \( x^2 \) or \( \sqrt{x} \). For the second type of calculator you have to press the \( \text{SHIFT} \) or \( 2\text{nd FUNCTION} \) key first.

Sometimes we need to answer questions such as,

"What number was squared to get 64?"

When answering this we need to use square roots.

The square root of a number is a number which, when squared (multiplied by itself), gives you the first number.

The sign \( \sqrt{\ } \) means square root.

We say that:

the square root of 64 is 8, i.e. \( \sqrt{64} = 8 \)

since the square of 8 is 64, i.e. \( 8^2 = 64 \)
The calculator button for finding a square root usually looks like $\sqrt{\phantom{0}}$. With some calculators you press the square root button before entering the number; with others you enter the number and then press the square root button. You need to find out how your calculator works out square roots.

Example 1
(a) Square each of these numbers:
   1, 5, 7, 14

(b) Find:
   \sqrt{25}, \sqrt{49}, \sqrt{196}, \sqrt{1}

Solution
(a) $1^2 = 1 \times 1 = 1$
   $5^2 = 5 \times 5 = 25$
   $7^2 = 7 \times 7 = 49$
   $14^2 = 14 \times 14 = 196$

(b) $\sqrt{25} = 5$ because $5^2 = 25$
    $\sqrt{49} = 7$ because $7^2 = 49$
    $\sqrt{196} = 14$ because $14^2 = 196$
    $\sqrt{1} = 1$ because $1^2 = 1$

Example 2
Use your calculator to find:
(a) $54^2$
(b) $\sqrt{961}$

Solution
(a) Either $5 \ 4 \ x^2 = 2916$ or $5 \ 4 \ \text{SHIFT} \ \sqrt{\phantom{0}} = 2916$

(b) Either $9 \ 6 \ 1 \ \sqrt{\phantom{0}} = 31$ or $\sqrt{\phantom{0}} \ 9 \ 6 \ 1 = 31$
Example 3

The area of this square is 225 cm². What is the length of each side?

**Solution**

Area $= (\text{length of side})^2$

$225 \text{ cm}^2 = (\text{length of side})^2$

Length of side $= \sqrt{225}$

$= 15 \text{ cm}$

Example 4

Use your calculator to find $\sqrt{5}$ correct to 2 decimal places.

**Solution**

$\sqrt{5} = 2.236067977$

$= 2.24$ correct to 2 decimal places.

Exercises

1. (a) Square these numbers:

   $2, 4, 9, 11, 12, 18, 20$

   (b) Use your answers to (a) to find:

   $\sqrt{144}, \sqrt{16}, \sqrt{121}, \sqrt{4}, \sqrt{81}, \sqrt{400}, \sqrt{324}$

2. Write down the following square roots *without* using a calculator:

   (a) $\sqrt{9}$  (b) $\sqrt{36}$  (c) $\sqrt{100}$

   (d) $\sqrt{169}$  (e) $\sqrt{225}$  (f) $\sqrt{0}$

3. Use a calculator to find these square roots, giving your answers correct to 2 decimal places:

   (a) $\sqrt{6}$  (b) $\sqrt{10}$  (c) $\sqrt{12}$

   (d) $\sqrt{20}$  (e) $\sqrt{50}$  (f) $\sqrt{90}$
4. What are the lengths of the sides of a square which has an area of 81 cm²?

5. A square has an area of 140 cm². How long are the sides of this square, to the nearest mm?

6. Explain why $7 < \sqrt{51} < 8$.

7. Copy the statements below and complete each one, putting two consecutive whole numbers in the empty spaces:
   
   (a) $\underline{} < \sqrt{70} < \underline{}$
   
   (b) $\underline{} < \sqrt{90} < \underline{}$

   (c) $\underline{} < \sqrt{5} < \underline{}$

   (d) $\underline{} < \sqrt{10} < \underline{}$

   (e) $\underline{} < \sqrt{115} < \underline{}$

   (f) $\underline{} < \sqrt{39} < \underline{}$

8. Decide whether each of these statements is true or false:
   
   (a) $4 < \sqrt{10} < 5$

   (b) $2.6 < \sqrt{7} < 2.7$

   (c) $3.4 < \sqrt{12} < 3.5$

   (d) $3.7 < \sqrt{15} < 3.8$

Write correct statements to replace those that are false, but keep the same square roots in them.

9. What is the perimeter of a square with area 196 cm²?

10. Three identical squares are put side-by-side to form a rectangle. The area of the rectangle is 192 cm². What are the lengths of the sides of the rectangle?
3 Pythagoras' Theorem

3.1 Pythagoras' Theorem

Pythagoras' Theorem relates the length of the hypotenuse of a right-angled triangle to the lengths of the other two sides.

The diagram opposite shows a right-angled triangle. The length of the hypotenuse is 5 cm and the other two sides have lengths 3 cm and 4 cm.

In this diagram, a square, A, has been drawn on the 3 cm side.

\[
\text{Area of square A} = 3 \times 3 = 9 \text{ cm}^2
\]

In this diagram, a second square, B, has been drawn on the 4 cm side.

\[
\text{Area of square B} = 4 \times 4 = 16 \text{ cm}^2
\]

Squares A and B together have total area:

\[
\text{Area A} + \text{Area B} = 9 + 16 = 25 \text{ cm}^2
\]

Finally, a third square, C, has been drawn on the 5 cm side.

\[
\text{Area of square C} = 5 \times 5 = 25 \text{ cm}^2
\]

We can see that

\[
\text{Area A} + \text{Area B} = \text{Area C}
\]

This formula is always true for right-angled triangles.
We now look at a right-angled triangle with sides $a$, $b$ and $c$, as shown opposite.

\[
\text{Area A } = a \times a = a^2 \\
\text{Area B } = b \times b = b^2 \\
\text{Area C } = c \times c = c^2
\]

So, \[\text{Area A } + \text{Area B } = \text{Area C}\]
gives us the formula
\[
a^2 + b^2 = c^2
\]
for all right-angled triangles.

Pythagoras' Theorem states that, for any right-angled triangle, the area of the square on the hypotenuse is equal to the sum of the areas of the squares on the two shorter sides.

If we use the letters $a$, $b$ and $c$ for the sides of a right-angled triangle, then Pythagoras' Theorem states that
\[a^2 + b^2 = c^2\]
where $c$ is the length of the hypotenuse.

**Example 1**

Verify Pythagoras' Theorem for the right-angled triangle opposite:
Solution

Here \( a = 9 \text{ cm}, \ b = 40 \text{ cm}, \ c = 41 \text{ cm}. \)

\[
a^2 = 9^2 = 9 \times 9 = 81
\]
\[
b^2 = 40^2 = 40 \times 40 = 1600
\]
\[
a^2 + b^2 = 1681
\]
\[
c^2 = 41^2 = 41 \times 41 = 1681
\]

So \( a^2 + b^2 = c^2 \) for this triangle.

Exercises

1. Which side is the *hypotenuse* in each of the following right angled triangles:

   (a) \( \triangle PQR \)

   (b) \( \triangle XYZ \)

   (c) \( \triangle JKL \)

   (d) \( \triangle RST \)

2. For each of the three diagrams at the top of the next page:

   (i) calculate the area of square A,

   (ii) calculate the area of square B,

   (iii) calculate the sum of area A and area B,

   (iv) calculate the area of square C,

   (v) check that:

      \[
      \text{area A} + \text{area B} = \text{area C}
      \]
3. Using the method shown in Example 1, verify Pythagoras' Theorem for the right-angled triangles below:

(a)  
(b)  
(c)  

4. The whole numbers 3, 4, 5 are called a Pythagorean triple because \(3^2 + 4^2 = 5^2\). A triangle with sides of lengths 3 cm, 4 cm and 5 cm is right-angled.

Use Pythagoras' Theorem to determine which of the sets of numbers below are Pythagorean triples:

(a) 15, 20, 25  
(b) 10, 24, 26  
(c) 11, 22, 30  
(d) 6, 8, 9
3.2 Calculating the Length of the Hypotenuse

Pythagoras' Theorem states that, for a right-angled triangle,

\[ c^2 = a^2 + b^2 \]

With this result it is very easy to calculate the length of the hypotenuse of a right-angled triangle.

**Example 1**

Calculate the length of the hypotenuse of a triangle in which the other two sides are of lengths 7 m and 8 m.

**Solution**

Let \( h \) be the length of the hypotenuse.

By Pythagoras' Theorem,

\[ h^2 = 8^2 + 7^2 \]
\[ h^2 = 64 + 49 \]
\[ h^2 = 113 \]
\[ h = \sqrt{113} \]
\[ h = 10.63014581 \text{ m} \]
\[ h = 10.6 \text{ m, correct to 1 decimal place} \]

**Example 2**

Calculate the length of the diagonals of the rectangle opposite:

**Solution**

The diagram shows the right-angled triangle that you need to use to find the length of the diagonal. The hypotenuse is the diagonal of the rectangle and this is labelled \( d \) on the diagram.
By Pythagoras’ Theorem,

\[ d^2 = 16^2 + 8^2 \]
\[ = 256 + 64 \]
\[ = 320 \]
\[ d = \sqrt{320} \]
\[ d = 17.88854382 \text{ cm} \]
\[ d = 17.9 \text{ cm, correct to 1 decimal place} \]

**Exercises**

1. Calculate the length of the hypotenuse of each of these triangles:

   (a)  
   \[
   \begin{array}{c}
   \text{8 cm} \\
   \text{6 cm}
   \end{array}
   \]

   (b)  
   \[
   \begin{array}{c}
   \text{15 mm} \\
   \text{3.6 mm}
   \end{array}
   \]

   (c)  
   \[
   \begin{array}{c}
   \text{40 cm} \\
   \text{9 cm}
   \end{array}
   \]

   (d)  
   \[
   \begin{array}{c}
   \text{16 cm} \\
   \text{30 cm}
   \end{array}
   \]

2. Calculate the length of the hypotenuse of each of the following triangles, giving your answers correct to 1 decimal place.

   (a)  
   \[
   \begin{array}{c}
   \text{8 cm} \\
   \text{10 cm}
   \end{array}
   \]

   (b)  
   \[
   \begin{array}{c}
   \text{5 cm} \\
   \text{8 cm}
   \end{array}
   \]

   (c)  
   \[
   \begin{array}{c}
   \text{6 cm} \\
   \text{5 cm}
   \end{array}
   \]

   (d)  
   \[
   \begin{array}{c}
   \text{4 m} \\
   \text{3.5 m}
   \end{array}
   \]
3. A rectangle has sides of lengths 5 cm and 10 cm. How long is the diagonal of the rectangle?

4. Calculate the length of the diagonal of a square with sides of length 6 cm.

5. The diagram shows a wooden frame that is to be part of the roof of a house:
   (a) Use Pythagoras' Theorem in triangle PQR to find the length PQ.
   (b) Calculate the length QS.
   (c) Calculate the total length of wood needed to make the frame.

6. An isosceles triangle has a base of length 4 cm and perpendicular height 8 cm. Giving your answers correct to 1 decimal place, calculate:
   (a) the length, \( x \) cm, of one of the equal sides,
   (b) the perimeter of the triangle.

7. One end of a rope is tied to the top of a vertical flagpole of height 5.2 m. When the rope is pulled tight, the other end is on the ground 3.8 m from the base of the flagpole. Calculate the length of the rope, giving your answer correct to 1 decimal place.

8. A rectangular lawn is 12.5 m long and 8 m wide. Matthew walks diagonally across the lawn from one corner to the other. He returns to the first corner by walking round the edge of the lawn. How much further does he walk on his return journey?

9. Which of the rectangles below has the longer diagonal?

   | 3 cm | 5 cm | Rectangle A |
   | 3.5 cm | 4.5 cm | Rectangle B |
3.2

10. (a) Use Pythagoras' Theorem to show that the length of the hypotenuse of this triangle is 10.0 cm correct to 1 decimal place.
(b) Maxine says that this triangle is isosceles because there are two sides of the same length.

Is Maxine correct?

3.3 Calculating the Lengths of Other Sides

Example 1
Calculate the length of the side marked $x$ in the following triangle:

Solution
By Pythagoras' Theorem:

\[ x^2 + 24^2 = 26^2 \]
\[ x^2 + 576 = 676 \]
\[ x^2 = 676 - 576 \]
\[ x^2 = 100 \]
\[ x = \sqrt{100} \]
\[ x = 10 \]

The length of the side $x$ is 10 cm.
Example 2
Calculate the perpendicular height of the isosceles triangle shown opposite:

Solution
The height can be calculated by using half of the original isosceles triangle, as shown:

The height has been labelled $h$ on the diagram.

By Pythagoras' Theorem:

\[ h^2 + 2^2 = 6^2 \]
\[ h^2 + 4 = 36 \]
\[ h^2 = 36 - 4 \]
\[ h^2 = 32 \]
\[ h = \sqrt{32} \]
\[ h = 5.656854249 \]

The perpendicular height of the triangle is 5.7 cm to 1 decimal place.

Exercises
1. Calculate the length of the side marked $x$ in each of the following triangles:

(a) \hspace{1cm}
(b) \hspace{1cm}
(c) \hspace{1cm}
(d) \hspace{1cm}
2. Calculate the length of the side marked \( x \) in each of the following triangles, giving your answer correct to 1 decimal place:

(a) \( \)  
(b) \( x \)

(c) \( \)  
(d) \( \)

3. Calculate the perpendicular height of this equilateral triangle, giving your answer correct to 1 decimal place.

4. Calculate the perpendicular height of an equilateral triangle with sides of length 5 cm, giving your answer correct to 1 decimal place.

5. Calculate the perpendicular height of the isosceles triangle shown opposite, giving your answer correct to 1 decimal place.

6. The width of a rectangle is 5 cm and the length of its diagonal is 13 cm.
   (a) How long is the other side of the rectangle?
   (b) What is the area of the rectangle?

7. The isosceles triangle at the top of the next page has 2 sides of length \( x \) cm. Copy and complete the calculation to find the value of \( x \) correct to 1 decimal place.
By Pythagoras' Theorem,
\[ x^2 + x^2 = 10^2 \]
\[ 2x^2 = 100 \]
\[ x^2 = \]
\[ x = \sqrt{50} \]
\[ x = \]
\[ x = \] to 1 decimal place.

8. The length of the diagonal of a square is 8 cm. How long are the sides of the square?

9. The diagram shows part of the framework of a roof.
   (a) Calculate the length XZ.
   (b) Calculate the length of YZ, correct to 1 decimal place.

10. A sheet is stretched over a washing line to make a tent, as shown in the diagram.
    (a) How high is the washing line above the ground? Give your answer to 1 decimal place.
    (b) If the same sheet was used and the washing line was now at a height of 1.25 m above the ground, what would be the width of the base of the tent? Give your answer correct to 1 decimal place.

11. A fishing rod is used to catch plastic ducks in a fairground game. The rod is 1 m long. A string with a ring is tied to the end of the rod. The length of the string is 0.4 m.
    When the ring is level with the lower end of the rod, as shown in the diagram, how far is the ring from that end of the fishing rod?
3.4 Problems in Context

When we use Pythagoras' Theorem to solve problems in context the first key step is to draw a right-angled triangle.

Example 1

A ladder is 5 m long. The bottom of the ladder is 2 m from the foot of a wall and the top leans against the wall. How high is the top of the ladder above the ground?

Solution

The first step is to draw a triangle to represent the situation. The height to the top of the ladder has been labelled $h$. (We assume that the ground is horizontal and the wall is vertical.)

Now use Pythagoras' Theorem:

\[ h^2 + 2^2 = 5^2 \]
\[ h^2 + 4 = 25 \]
\[ h^2 = 25 - 4 = 21 \]
\[ h = \sqrt{21} \]
\[ h = 4.582575695 \]

The top of the ladder is 4.58 m above the ground (to the nearest cm).

Example 2

A ship sails 300 km due west and then 100 km due south. At the end of this journey, how far is the ship from its starting position?

Solution

The first step is to draw a diagram showing the ship's journey. The distance from the starting point has been labelled $d$.

Now use Pythagoras Theorem:

\[ d^2 = 300^2 + 100^2 \]
\[ d^2 = 90000 + 10000 \]
\[ d^2 = 100,000 \]
\[ d = \sqrt{100000} \]
\[ d = 316.227766 \]

The distance from the starting point is 316 km to the nearest km.

**Example 3**

Calculate the area of the triangle shown opposite:

**Solution**

The length of the unknown side has been marked \( x \).

Using Pythagoras’ Theorem,

\[
x^2 + 6^2 = 10^2
\]
\[
x^2 + 36 = 100
\]
\[
x^2 = 100 - 36
\]
\[
x^2 = 64
\]
\[
x = \sqrt{64}
\]
\[
x = 8 \text{ cm}
\]

Area of the triangle = \[
\frac{1}{2} \times \text{base} \times \text{perpendicular height}
\]
\[
= \frac{1}{2} \times 6 \times 8
\]
\[
= 24 \text{ cm}^2
\]

**Exercises**

1. A hiker walks 300 m due north and then 400 m due east. How far is the hiker now from her starting position?

2. A ladder of length 4 m leans against a wall so that the top of the ladder is 3 m above ground level. How far is the bottom of the ladder from the wall?

3. Two remote-controlled cars set off from the same position. After a short time one has travelled 20 m due north and the other 15 m due east. How far apart are the two cars?
4. A room should have a rectangular floor, with sides of lengths 4 m and 5 m. A builder wants to check that the room is a perfect rectangle and measures the two diagonals of the room, which should be the same length. To the nearest cm, how long should each diagonal be?

5. For the triangle shown opposite,
   (a) calculate the length $x$,
   (b) calculate the area of the triangle.

6. Calculate the perimeter of the triangle shown opposite, giving your answer correct to 1 decimal place.

7. Calculate the perimeter of the parallelogram below, giving your answer to the nearest millimetre.

8. One end of a rope of length 10 m is tied to the top of a vertical flag pole. When the rope is tight it can touch the ground at a distance of 4 m from the base of the pole. How tall is the flagpole? Give your answer correct to the nearest cm.

9. A guy rope on a tent is 1.5 m long. One end is fixed to the top of a vertical pole and the other is pegged to the ground. If the pole is 1.2 m high, how far is the pegged end of the rope from the base of the flagpole?

10. Ron's dad says that Ron must not walk on the lawn. The lawn is a rectangle with sides of lengths 10 m and 16 m. When his dad is looking, Ron walks from his house to the gate by walking along two edges of the lawn. When his dad is not looking, Ron walks diagonally across the lawn. How much further does Ron have to walk to get from the house to the gate when his dad is looking? Give your answer to a suitable level of accuracy.
3.5 Constructions and Angles

The formula for Pythagoras' Theorem can be used to decide if a triangle is right-angled.

In any triangle,

*the longest side faces the largest angle*

*the shortest side faces the smallest angle.*

In a triangle with longest side $c$, and other two sides $a$ and $b$,

- if $c^2 = a^2 + b^2$, then the angle opposite $c = 90^\circ$;
- if $c^2 < a^2 + b^2$, then the angle opposite $c < 90^\circ$ (so all three angles are acute);
- if $c^2 > a^2 + b^2$, then the angle opposite $c > 90^\circ$ (i.e. the triangle has an obtuse angle).

Example 1

(a) Use a ruler and a pair of compasses to construct this triangle:

![Triangle with sides 3 cm, 5 cm, and 4 cm](image)

(b) Use a protractor to check that the triangle has a right angle.

(c) Confirm that Pythagoras' Theorem is true for this triangle.

Solution

(a) First draw a line with length 4 cm.

Then draw an arc of radius 3 cm with centre on the left-hand end of the line.

Next draw an arc of radius 5 cm with centre on the right-hand end of the line.

The point where the two arcs cross is the third corner of the triangle.
(b) The angle at the bottom left-hand corner measures 90°, so the triangle has a right angle.

(c) Here \( a = 4 \text{ cm} \), \( b = 3 \text{ cm} \) and \( c = 5 \text{ cm} \).

\[
\begin{align*}
  a^2 + b^2 &= 4^2 + 3^2 \\
  &= 16 + 9 \\
  &= 25
\end{align*}
\]

\[
\begin{align*}
  c^2 &= 5^2 \\
  &= 25
\end{align*}
\]

Therefore \( a^2 + b^2 = c^2 \)

So Pythagoras' Theorem is true in this case, confirming that this is a right-angled triangle.

Example 2

Which of these triangles contains a right angle?

![Diagram of triangles](image)

Solution

We use Pythagoras' Theorem to find out if a triangle is right-angled, using \( c \) for the longest side.

(a) In this triangle, \( a = 5 \), \( b = 12 \) and \( c = 13 \).

\[
\begin{align*}
  a^2 + b^2 &= 5^2 + 12^2 \\
  &= 25 + 144 \\
  &= 169
\end{align*}
\]

Here \( a^2 + b^2 = c^2 \), so this triangle does contain a right angle.
(b) In this triangle,  \( a = 6, \ b = 7 \) and \( c = 8 \).

\[
\begin{align*}
a^2 + b^2 &= 6^2 + 7^2 \\
&= 36 + 49 \\
&= 85
\end{align*}
\]

Here \( c^2 \neq a^2 + b^2 \), so the triangle does not contain a right angle. As \( c^2 < a^2 + b^2 \), the angle opposite \( c \) is less than 90°, so all the angles in this triangle are acute.

(c) Here \( a = 6, \ b = 11 \) and \( c = 14 \).

\[
\begin{align*}
a^2 + b^2 &= 6^2 + 11^2 \\
&= 36 + 121 \\
&= 157
\end{align*}
\]

Here \( c^2 \neq a^2 + b^2 \), so the triangle does not contain a right angle. As \( c^2 > a^2 + b^2 \) the angle opposite \( c \) is greater than 90°, so the triangle contains one obtuse angle.

**Exercises**

1. (a) Using a ruler and a pair of compasses, construct a triangle with sides of lengths 6 cm, 8 cm and 10 cm.

(b) Use a protractor to measure the angles of your triangle.

(c) Is the triangle right-angled?

(d) Use Pythagoras' Theorem to decide whether the triangle is right-angled.

(e) Was your answer to part (c) correct?

2. Repeat question 2 for a triangle with sides of lengths 7 cm, 8 cm and 11 cm.

3. Decide which of the triangles described below:

(a) is right-angled,

(b) contains an obtuse angle,

(c) contains all acute angles.

In each case, show how you reached your conclusion.

(i) a triangle with sides of lengths 10 cm, 11 cm and 14 cm

(ii) a triangle with sides of lengths 10 cm, 12 cm and 16 cm

(iii) a triangle with sides of lengths 9 cm, 12 cm and 15 cm
4. (a) Use an accurate construction to find out if a triangle with sides of lengths 6 cm, 7 cm and 12 cm contains a right angle.

(b) Use Pythagoras' Theorem to check your answer to part (a).

5. Ahmed draws a square with sides of length 6 cm. He then measures a diagonal as 8.2 cm. Use Pythagoras' Theorem to decide if Ahmed has drawn the square accurately.

6. An isosceles triangle has 2 sides of length 8 cm. The length of the base is 9 cm. Decide, by calculation, whether the angle $\theta$ is a right angle, an acute angle or an obtuse angle. Show clearly how you reached your conclusion.

7. Measure the lengths of the sides and diagonal of your textbook. Use your measurements to decide whether the corners of your book are right-angled.

8. A triangle has sides of lengths 21 cm, 28 cm and $x$ cm.

(a) Show that the triangle has a right angle if $x = 35$.

(b) For what values of $x$ will the triangle contain an obtuse angle?

9. An isosceles triangle is known to have one side of length 18 cm and one side of length 28 cm.

(a) Explain why the triangle cannot contain a right angle,

(b) Show, by calculation, that it is possible for the triangle to contain three acute angles. Draw a sketch of the triangle in this case.

10. A right-angled triangle has two sides of lengths 24 cm and 32 cm. Use Pythagoras' Theorem to calculate the length of the other side. [Note: there are 2 possible answers.]
4 Rounding and Estimating

4.1 Revision of the Four Rules: Whole Numbers

Example 1

Calculate:
(a) 464 + 97  
(b) 184 − 36  
(c) 47 × 12  
(d) 710 ÷ 5

Solution

(a) 464
+ 97
561
(b) 184
− 36
148
(c) 47
× 12
564
(d) 710
÷ 5
142

Exercises

Work out the answer to each question without using a calculator. Check your answers with a calculator.

1. (a) 13 + 16  
(b) 24 + 22  
(c) 45 + 34  
(d) 123 + 51  
(e) 214 + 135  
(f) 201 + 356

2. (a) 36 + 102  
(b) 88 + 35  
(c) 66 + 282  
(d) 97 + 142  
(e) 361 + 421  
(f) 188 + 924

3. (a) 25 − 13  
(b) 66 − 22  
(c) 97 − 46  
(d) 136 − 121  
(e) 258 − 39  
(f) 971 − 420
4.1

4. (a) $199 - 42$  
(b) $643 - 132$  
(c) $198 - 156$  
(d) $372 - 184$  
(e) $924 - 138$  
(f) $3631 - 179$

5. (a) $12 \times 3$  
(b) $11 \times 5$  
(c) $23 \times 2$  
(d) $31 \times 3$  
(e) $22 \times 4$  
(f) $101 \times 6$

6. (a) $19 \times 5$  
(b) $86 \times 4$  
(c) $39 \times 6$  
(d) $27 \times 7$  
(e) $43 \times 9$  
(f) $65 \times 8$

7. (a) $82 \times 11$  
(b) $37 \times 12$  
(c) $39 \times 42$  
(d) $54 \times 23$  
(e) $61 \times 34$  
(f) $87 \times 65$

8. (a) $68 \div 2$  
(b) $64 \div 4$  
(c) $123 \div 3$  
(d) $845 \div 5$  
(e) $312 \div 6$  
(f) $1407 \div 7$

9. (a) $240 \div 20$  
(b) $720 \div 12$  
(c) $880 \div 44$  
(d) $630 \div 15$  
(e) $750 \div 25$  
(f) $345 \div 23$

10. (a) $87 \times 3$  
(b) $192 + 249$  
(c) $186 - 95$  
(d) $36 \times 43$  
(e) $915 \div 5$  
(f) $48 \times 17$

4.2 Revision of the Four Rules: Decimals

Example 1
Calculate:

(a) $3.8 + 10.42$  
(b) $18.2 - 0.36$  
(c) $8.2 \times 3.7$  
(d) $1.56 \div 0.3$

Solution

(a) \[
\begin{array}{c}
3.8 \\
+ 10.42 \\
\hline
14.22
\end{array}
\]

(b) \[
\begin{array}{c}
18.2 \\
- 0.36 \\
\hline
17.84
\end{array}
\]

(c) \[
\begin{array}{c}
7 \\
+ 1.2 \\
\hline
8.2
\end{array}
\]

(d) \[
\begin{array}{c}
1.56 \\
\div 0.3 \\
\hline
5.2
\end{array}
\]

64
(c) \[ 8.2 \times 3.7 = 5.2 \]
\[ 5.74 \]
\[ 2460 \]
\[ 30.34 \]
\[ 1 \]

Exercises

Solve each of the following \textit{without} using a calculator. Check your answers with a calculator.

1. (a) \[ 3.5 + 4.2 \]
   (b) \[ 16.1 + 32.6 \]
   (c) \[ 1.5 + 3.8 \]
   (d) \[ 13.3 + 4.61 \]
   (e) \[ 18.6 + 0.42 \]
   (f) \[ 3.14 + 0.612 \]

2. (a) \[ 6.4 - 2.1 \]
   (b) \[ 27.8 - 13.6 \]
   (c) \[ 3.2 - 0.8 \]
   (d) \[ 8.2 - 4.5 \]
   (e) \[ 6.62 - 0.34 \]
   (f) \[ 8.3 - 6.27 \]

3. (a) \[ 4.3 \times 2 \]
   (b) \[ 3.5 \times 4 \]
   (c) \[ 7.4 \times 6 \]
   (d) \[ 6.2 \times 7 \]
   (e) \[ 18.3 \times 9 \]
   (f) \[ 5.62 \times 5 \]

4. (a) \[ 6.8 \div 2 \]
   (b) \[ 63.9 \div 3 \]
   (c) \[ 52.4 \div 4 \]
   (d) \[ 75.5 \div 5 \]
   (e) \[ 99.4 \div 7 \]
   (f) \[ 151.8 \div 6 \]

5. (a) \[ 12.6 + 8.5 \]
   (b) \[ 76.3 - 18.7 \]
   (c) \[ 20.39 - 15.6 \]
   (d) \[ 17.6 \times 4 \]
   (e) \[ 132.7 \times 6 \]
   (f) \[ 36.61 \div 7 \]

6. (a) \[ 5.6 \times 0.3 \]
   (b) \[ 2.3 \times 1.5 \]
   (c) \[ 4.8 \times 0.21 \]
   (d) \[ 3.4 \times 9.4 \]
   (e) \[ 3.6 \times 0.72 \]
   (f) \[ 8.2 \times 0.91 \]

7. (a) \[ 18.6 \div 0.3 \]
   (b) \[ 74.5 \div 0.5 \]
   (c) \[ 0.36 \div 0.02 \]
   (d) \[ 10.5 \div 5 \]
   (e) \[ 45 \div 0.09 \]
   (f) \[ 0.84 \div 0.4 \]

8. (a) \[ 21.6 \div 0.4 \]
   (b) \[ 8.2 - 0.37 \]
   (c) \[ 0.62 \times 7 \]
   (d) \[ 3.2 \times 0.17 \]
   (e) \[ 8.4 \div 8 \]
   (f) \[ 3.7 \times 2.01 \]
### 4.3 Order of Operations

B rackets  
O    
D ivision  
M ultiplication  
A ddition  
S ubtraction  

**BODMAS** can be used to remember the order in which to carry out operations  

#### Example 1

Calculate:

(a) $32 + (6 \times 8)$  
(b) $4 \times 6 + 18 \div 2$  
(c) $(17 - 2) \div 5 + 6$

**Solution**

(a) $32 + (6 \times 8)$  
\[
\begin{align*}
&= 32 + (48) \\
&= 80
\end{align*}
\]

(b) $4 \times 6 + 18 \div 2$  
\[
\begin{align*}
&= 24 + 9 \\
&= 33
\end{align*}
\]

(c) $(17 - 2) \div 5 + 6$  
\[
\begin{align*}
&= 15 \div 5 + 6 \\
&= 3 + 6 \\
&= 9
\end{align*}
\]

#### Example 2

State whether each one of the statements below is *true* or *false*:

(a) $3 + 6 \times 2 = 15$  
(b) $30 - 7 \times 4 = 92$  
(c) $8 + 20 \div 2 = 14$
Solution

(a) \[3 + 6 \times 2 = 3 + 12\]
\[= 15\]

Therefore the statement is true.

(b) \[30 - 7 \times 4 = 30 - 28\]
\[= 2\]

Therefore the statement is false.

(c) \[8 + 20 \div 2 = 8 + 10\]
\[= 18\]

Therefore the statement is false.

Exercises

1. Calculate:
   (a) \[6 + 7 \times 2\]
   (b) \[8 - 3 \times 2\]
   (c) \[19 - 4 \times 3\]
   (d) \[3 \times 6 - 9\]
   (e) \[15 - 4 + 7 \times 2\]
   (f) \[11 \times 3 + 2\]
   (g) \[16 \times 4 - 3\]
   (h) \[6 + 7 \times 2 - 20 \div 4\]
   (i) \[18 \times 2 - (4 + 7)\]
   (j) \[16 - 5 \times 2 + 3\]

2. State whether each one of the statements below is true or false. Calculate the correct answer for those that are false.
   (a) \[6 \times 7 - 2 = 40\]
   (b) \[8 \times (6 - 2) + 3 = 56\]
   (c) \[35 - 7 \times 2 = 56\]
   (d) \[3 + 7 \times 3 = 30\]
   (e) \[18 - (4 + 7) = 21\]
   (f) \[43 - 3 + 2 = 42\]
   (g) \[80 \div 2 + 6 = 10\]
   (h) \[64 - 10 + 2 = 52\]

3. Put brackets into each of the statements below to make it correct:
   (a) \[3 \times 6 + 1 = 21\]
   (b) \[5 + 6 \times 2 = 22\]
   (c) \[45 \div 6 + 3 = 5\]
   (d) \[49 - 3 + 2 = 44\]
   (e) \[7 \times 3 + 2 = 35\]
   (f) \[13 - 4 \times 2 = 18\]
4. Write out each of the calculations below, filling in the missing numbers:
   (a) \(3 \times ? + 2 = 17\)   (b) \(? \times 5 - 8 = 22\)
   (c) \((4 + ?) \times 2 = 20\)   (d) \(6 - ? \times 2 = 0\)
   (e) \((7 - ?) \times 4 = 20\)   (f) \(? \div 3 + 4 = 8\)

5. Jane writes down:
\[4 \times 7 + 2 \times 3 = 90\]
   (a) Explain why her answer is incorrect, and calculate the correct answer.
   (b) By using brackets Jane can make her calculation correct. Show how this can be done.

6. Esther and Andy are given this problem:
\[30 \div 6 - 3 + 1\]
   Esther says the answer is 1.
   Andy says the answer is 11.
   (a) Is either of them correct?
   (b) Show how Esther could insert brackets to give her answer.
   (c) Show how Andy could insert brackets to give his answer.

7. State whether each one of the statements below is true or false:
   (a) \((3 \times 6) \times 2 = 3 \times (6 \times 2)\)
   (b) \((4 + 2) + 7 = 4 + (2 + 7)\)
   (c) \((8 - 2) - 1 = 8 - (2 - 1)\)
   (d) \((8 \div 2) \div 2 = 8 \div (2 \div 2)\)

8. Put brackets into each of the calculations below to make it correct:
   (a) \(13 - 4 - 1 = 10\)
   (b) \(30 - 9 + 2 = 19\)
   (c) \(60 \div 6 \div 3 = 30\)

9. Calculate:
   (a) \(8.2 \div 0.2 - 0.1\)
   (b) \(3.6 \times 0.2 - 0.1\)
   (c) \(8.2 \times (6 - 5.4)\)
   (d) \(2.2 - 0.7 \times 0.2\)

10. Write out each of the calculations below, filling in the missing numbers:
   (a) \(0.8 + ? \times 0.6 = 3.2\)
   (b) \(? \times 0.5 + 6 \times 0.4 = 3.9\)
   (c) \(0.9 + 4.8 \div ? = 6.9\)
   (d) \(2.7 \div ? - 1.4 = 1.6\)
4.4 Problems in Context

Example 1

Packets of football stickers cost 32p each. Calculate the total cost of 25 packets of stickers.

Solution

Working in pence, total cost is

\[
\begin{array}{c}
32 \\
\times 25 \\
160 \\
640 \\
800p \\
\end{array}
\]

Hence the total cost is £8.00.

Example 2

Tickets for a concert cost £8 each. Rebecca has £50 to spend on tickets for the concert. How many of her friends can she buy tickets for and how much money does she have left from her £50?

Solution

\[
\begin{array}{c}
6 \\
\text{Remainder 2} \\
8 \overline{50} \\
\end{array}
\]

So Rebecca can buy 6 tickets, one for herself and one each for 5 friends. She will have £2 left.

Example 3

A taxi driver charges his passengers £1.25 plus 64p per mile. Calculate the cost of:

(a) a 10 mile journey, 

(b) a 3 mile journey.

Solution

Working in pounds, total cost is:

(a) \[1.25 + 10 \times 0.64 = 1.25 + 6.40 = £7.65\]

(b) \[1.25 + 3 \times 0.64 = 1.25 + 1.92 = £3.17\]
Exercises

_In these questions, do not use a calculator, and remember to show all of your working._

1. Tickets for a school party cost £1.25 each. Calculate the cost of:
   (a) 3 tickets,  
   (b) 14 tickets.

2. CDs cost £9 each in a music shop sale. How many CDs can you buy if you have £48 to spend? How much money will you have left over?

3. A school buys 30 calculators costing £6.99 each. What is the total cost of these calculators? How could you do this calculation in your head?

4. A school mathematics department has £300 to spend on new textbooks. The textbooks cost £7 each. How many books can be bought?

5. Prakesh is paid travelling expenses every time he drives his car for work. He is paid £12 for each journey, plus 14p per mile travelled. How much is he paid for:
   (a) a 50 mile journey,  
   (b) an 82 mile journey?

6. How many minibuses, each seating 17 pupils are needed to transport 110 pupils?

7. Joanne buys 3 magazines that cost £1.50, £2.45 and 80p. She pays for them with a £10 note. How much change should she get?

8. Ben orders 25 floppy discs for his computer. The discs cost 40p each and he has to pay £3.25 postage. How much does he have to pay in total?

9. A farmer packs his free-range eggs into boxes that each contain half a dozen eggs. One day he collects 119 eggs. How many boxes can he fill and how many eggs does he have left over?

10. Alison buys 6 tapes that cost £8.99 each. She pays for them with three £20 notes. How much change should she get?

4.5 Rounding

We _round_ numbers when all we need is a _reasonable approximation_ rather than the exact value.

The number 8.4236 can be rounded to a specified number of _decimal places_:

- $8.4236 \rightarrow 8.424$ to 3 decimal places
- $8.4236 \rightarrow 8.42$ to 2 decimal places
- $8.4236 \rightarrow 8.4$ to 1 decimal place
The number 173.265 can be rounded to a specified number of significant figures:
\[
\begin{align*}
173.265 & \rightarrow 173.27 \text{ to 5 significant figures} \\
173.265 & \rightarrow 173 \text{ to 3 significant figures} \\
173.265 & \rightarrow 200 \text{ to 1 significant figure}
\end{align*}
\]

A particular digit in a number will \textit{round up} if the digit that follows it is 5, 6, 7, 8 or 9.

For example, \( \frac{52.368}{1} = 52.37 \) (to 2 decimal places).

The digit will \textit{remain unchanged} if the following digit is 0, 1, 2, 3 or 4.

For example, \( \frac{6.743}{1} = 6.74 \) (to 3 significant figures).

**Example 1**

Round 3647.5 to the nearest:

(a) whole number, 
(b) ten (10), 
(c) hundred (100), 
(d) thousand (1000).

**Solution**

(a) 3648 Note that the 7 rounds up to 8, because it is followed by a 5.

(b) 3650 Note that the 4 rounds up to a 5, because it is followed by a 7.

(c) 3600 Note that the 6 is unchanged, because the digit following it in the number is less than 5.

(d) 4000 Note that the 3 rounds up to a 4, because it is followed by a 6.

**Example 2**

Write 13.68952 correct to:

(a) 1 decimal place, 
(b) 3 decimal places, 
(c) 2 decimal places.

**Solution**

(a) 13.7 to 1 decimal place.

(b) 13.690 to 3 decimal places.

(c) 13.69 to 2 decimal places.
Example 3
Write:
(a) 3.642 correct to 2 significant figures,
(b) 314 269 correct to 3 significant figures,
(c) 0.00723 correct to 1 significant figure.

Solution
(a) 3.6 correct to 2 significant figures.  (Note the need for zeros to replace
the remaining digits of the original
number, to give a rounded number
of comparable size.)
(b) 314 000 correct to 3 significant figures.  (Note that the zeros before the 7 are not
significant, so they are not counted. The 7 is
the first significant figure, i.e. it is the first
digit that really determines the size of the
number.)
(c) 0.007 to 1 significant figure.  (Note that the zeros before the 7 are not
significant, so they are not counted. The 7 is
the first significant figure, i.e. it is the first
digit that really determines the size of the
number.)

Exercises
1. Round each of the numbers below to the nearest whole number:
   (a) 4.3   (b) 2.04   (c) 16.9
   (d) 3.5   (e) 33.49  (f) 18.65

2. Round each of these numbers to the nearest ten:
   (a) 187   (b) 309   (c) 8
   (d) 35    (e) 44.9  (f) 16.4

3. The attendance at a football match was 36 475 people.
   Round this number to:
   (a) the nearest 1000  (b) the nearest 100  (c) the nearest 10.

4. Write each of these numbers correct to 2 decimal places:
   (a) 4.263   (b) 0.0472  (c) 10.8374
   (d) 82.062   (e) 3.445   (f) 9.395

5. Write each of these numbers correct to 2 significant figures:
   (a) 1.473   (b) 6.254   (c) 3.216
   (d) 10.68   (e) 142     (f) 1374
6. Write the number 8.645712 correct to:
   (a) 3 decimal places  (b) 4 decimal places  (c) 1 decimal place.

7. Write the number 147.52 correct to:
   (a) 4 significant figures,  (b) 3 significant figures,
   (c) 2 significant figures,  (d) 1 significant figure.

8. Write the number 104.735 correct to:
   (a) the nearest whole number,  (b) 2 decimal places,
   (c) 2 significant figures,  (d) 1 decimal place,
   (e) 1 significant figure.

9. Write each of the numbers below correct to 3 significant figures:
   (a) 18.47  (b) 0.003265  (c) 147 300
   (d) 62.999  (e) 0.036247  (f) 0.00036945

10. A student completes the table opposite, but puts the accuracy statements against the wrong numbers.
    Copy the table and put the statements against the numbers so that every pairing is correct.

<table>
<thead>
<tr>
<th>Number</th>
<th>Accuracy</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.047</td>
<td>4 significant figures</td>
</tr>
<tr>
<td>0.003</td>
<td>2 significant figures</td>
</tr>
<tr>
<td>16.22</td>
<td>3 significant figures</td>
</tr>
<tr>
<td>184 200</td>
<td>2 decimal places</td>
</tr>
<tr>
<td>7.06</td>
<td>3 decimal places</td>
</tr>
</tbody>
</table>

4.6 Estimating

We can *estimate* the answers to calculations by rounding all the numbers sensibly. We often round to just one significant figure. However, depending on the numbers involved in the calculation, it may be better to round sensibly than to one significant figure.

For example, \( 33.78 \div 17.24 \) is roughly \( 34 \div 17 \), when the numbers are rounded sensibly, giving a simple estimate for the answer of 2.

Example 1

A box of chocolates costs £2.72.

Estimate the total cost of 4 boxes of chocolates.

**Solution**

(a) Cost (£) = 4 \times 2.72

Estimate = 4 \times 3

= £12
To make sure that you obtain the correct answer when you use a calculator,

- **ESTIMATE** the answer mentally,
- **CALCULATE** the answer using your calculator and then
- **CHECK** that the calculator answer is sensible by comparing it with your mental estimate.

**Example 2**

Halim uses his calculator to work out $8.623 \times 4.71$.

He gets the answer 406.1433.

Use an estimate to check his answer.

**Solution**

(a) Estimate $= 9 \times 5$

$= 45$

Halim's answer should have been 40.61433.

**Example 3**

Jai carries out the following calculations on his calculator, and writes his answers correct to 3 decimal places:

A $3.62 \times 8.94 = 32.363$

B $47.92 \div 2.17 = 1.512$

C $184 \times 3.616 = 665.344$

D $(21.4 + 19.7) \times 3.61 = 14.837$

Use estimates to decide which answers could be correct and which are definitely incorrect.

**Solution**

A Estimate $= 4 \times 9$

$= 36$

suggested that Jai’s answer could be correct.

B Estimate $= 50 \div 2$

$= 25$

showing that Jai’s answer must be incorrect.
C Estimate  =  200 × 4  
      =  800  
suggesting that Jai's answer could be correct.

D Estimate  =  (20 + 20) × 4  
      =  40 × 4  
      =  160  
showing that Jai's answer must be incorrect.

Exercises

1. For each of the calculations listed below,
   (i) estimate the answer,
   (ii) use a calculator to work out the answer,
   (iii) compare your estimate with the answer from the calculator:

   (a)  4.7 × 8.34  
   (b)  9.6 × 21.43  
   (c)  11.46 × 8.02  
   (d)  18.3 × 108  
   (e)  95 × 76  
   (f)  15.4 × 24.9

2. Boxes of matches each contain 52 matches. Estimate the total number of matches in 8 boxes.

3. The floor of a room measures 3.61 m by 4.72 m.
   (a) Estimate the area of the floor of the room.
   (b) Calculate the area of the floor, using a calculator.
   (c) Compare your estimate with the answer from the calculator.

4. Estimate the cost of 23 cans of drink costing 37p each.

5. Kyle uses his calculator to do the calculations listed below, and gives his answers correct to 3 decimal places:

   A  36.41 × 37.32 = 135.882  
   B  56.2 × 1.97 = 11.071  
   C  82.3 × 0.625 = 51.438  
   D  (204 + 109) × 10.2 = 3.193  
   E  (16.7 + 31.3) ÷ 4.75 = 1.011

By using estimates, decide which calculations Kyle has not done correctly.
6. Make estimates for each of the calculations below:
   (a) \( \frac{6.1 \times 3.4}{4.2} \)
   (b) \( \frac{7.3 + 9.1}{2.3} \)
   (c) \( \frac{62.6 \times 21.3}{34.9} \)
   (d) \( \frac{71.3 \times 99.6}{11.3} \)
   (e) \( \frac{142.3 - 93.6}{23.8} \)
   (f) \( \frac{16.5 \times 19.2}{33.6 - 21.9} \)

7. Estimate the cost of 18 calculators that cost £7.99 each.

8. A can contains 330 ml of a drink and there are 144 cans in a box. Estimate the total volume of drink in a box, in litres.

9. A car uses 0.18 litres of fuel to travel 1 mile.
   (a) Estimate the amount of fuel that is used on a 162 mile journey.
   (b) Use a calculator to work out the amount of fuel that is used.
   (c) Does your estimate support the answer from your calculator?

10. Tickets to watch a football match cost £19 each. If 26,472 people watch the match, estimate the total amount that has been paid by these spectators.

4.7 Calculator Logic - Bracket and Memory Keys

When using your calculator it is important to be aware of both how it works and how to make the best use of it. Most calculators have bracket and memory keys that can be used for more complex calculations.

Note: A scientific calculator will always try to apply the rules of BODMAS.

Brackets can be inserted at any stage of a calculation by using the bracket keys { and } . The calculator may show an error message if brackets are not in pairs.

The notation on calculator memory keys varies from one machine to another. You need to find out the keys on your calculator that perform the following functions:

- \( \text{M in} \) or \( \text{STO} \) places a number on display in the memory.
- \( \text{M +} \) adds the number on display to the number in the memory.
Some of these keys perform other functions in other modes (especially in statistical mode).

One thing you will need to find out for yourself is how to empty the contents of the memory; this varies from one calculator to another.

**Example 1**

Calculate

\[
\frac{6.2 + 8.6}{3.9 - 2.4}
\]

correct to 3 significant figures.

**Solution**

Using brackets,

\[
(6.2 + 8.6) \div (3.9 - 2.4) = \text{gives 9.87 to 3 significant figures}
\]

Using the memory,

\[
3.9 \ \text{MR} \ \text{M in}
\]

\[
6.2 + 8.6 \div 3.9 = \text{MR} = \text{gives 9.87 to 3 significant figures}
\]

**Example 2**

Calculate

\[
\frac{6}{3 + 4 \times 7.2}
\]

correct to 2 decimal places.

**Solution**

Using brackets,

\[
6 \div (3 + 4 \times 7.2) = \text{gives 0.1886 . . . = 0.19 to 2 decimal places}
\]

*(Remember that a scientific calculator will apply BODMAS.)*
Using memory,

\[
\begin{align*}
3 & \;+\; 4 \;\times\; 7.2 \;=\; \text{Min} \\
6 & \;\div\; \text{MR} \;=\; \text{gives} \; 0.1886 \ldots \\
& \;=\; 0.19 \;\text{to} \; 2 \;\text{decimal places}
\end{align*}
\]

Example 3
Do you need to include brackets if you use a scientific calculator to work out:

(a) \(3 \times 4 + 6 \times 2\)  
(b) \(\frac{24}{8 - 2}\)

Solution
(a) The correct answer is \(3 \times 4 + 6 \times 2 = 12 + 12\)

\[= 24\]

Using a scientific calculator without brackets also gives 24, so brackets are not needed here.

(b) The correct answer is

\[
\frac{24}{8 - 2} = \frac{24}{6} = 4
\]

Without brackets the calculator gives the answer 1.

It actually works out \(24 \div 8 - 2\) or \(\frac{24}{8} - 2\) which gives 1, so brackets are needed here.

Exercises
1. Carry out the following calculations using a calculator, giving your answers, where necessary, correct to 2 decimal places:

(a) \(6 \times (8.7 - 1.05)\)  
(b) \(\frac{2 \times 47}{6 + 9}\)

(c) \(\frac{6 + 17}{3}\)  
(d) \(\frac{42 - 3}{7}\)

(e) \(\frac{6 + 22}{47 - 21}\)  
(f) \(\frac{9 - 32}{8 - 27}\)
2. Carry out the calculation below. You may use the memory of your calculator, but not the bracket keys. Give your answers correct to 3 significant figures.

(a) \( \frac{4.9}{3.7 \times 2.6} \)  
(b) \( \frac{4.7}{16 - 7} \)  
(c) \( \frac{9.2 \times 6.7}{4 + 16.2} \)  
(d) \( \frac{11.2 - 9.47}{12 - 0.81} \)

3. Use brackets or the memory facilities on a calculator to calculate the following, giving your answers, where necessary, correct to 3 decimal places.

(a) \( 8 + \frac{6}{9 + 7} \)  
(b) \( \frac{1.9 + 12.2}{8 - 3} \)  
(c) \( \frac{6.3 \times 5.32 + 6.49}{(2.94 - 1.62) \times 3.5} \)  
(d) \( \frac{21.5 + 6.7 + 3.2}{8 + 3} \)

4. Calculate

\( \frac{4.7 \times (5.32 + 6.49)}{(2.94 - 1.62) \times 3.5} \)

(a) correct to 2 significant figures,  
(b) correct to 2 decimal places.

5. James tried to calculate \( \frac{6}{8 + 2} \). He obtained the answer 2.75, which is wrong.

(a) What is the correct answer?  
(b) What did James do wrong?

6. Do you need to use brackets if you work the calculations below out on a calculator, without using the memory facilities?

(a) \( \frac{6 + 9}{2} \)  
(b) \( \frac{8 + 3 \times 6}{4} \)  
(c) \( \frac{6 + 9}{2} \)  
(d) \( \frac{8 + 3 \times 6}{4} \)

7. Calculate \( \frac{3 + 9 + 17 + 8 + 6 + 9 + 4 + 7}{5 + 3} \) using a calculator, giving your answer correct to:

(a) 2 decimal places,  
(b) 2 significant figures.
8. For each set of instructions given below, write down the calculation that it was used to find:

(a) \[ 6 \div (8 - 7 + 2) = \]

(b) \[ 4 + 7 \times (9 + 4) = \]

(c) \[ 4 \div (8 - 5) + 6 = \]

(d) \[ 1 + 7 \times (8 - 3) \div (2 + 9) = \]

9. Without using the fraction key, use your calculator to work out the following, giving your answers, where necessary, correct to 3 significant figures:

(a) \[ \frac{1}{6} + \frac{1}{5} + \frac{1}{4} + \frac{1}{3} + 1 = \]

(b) \[ \frac{1}{6} + \frac{1}{5} + \frac{1}{4} + \frac{1}{3} = \]

(c) \[ 1 + \left( \frac{1}{6 + \frac{1}{5}} \right) = \]
5 Data Analysis

5.1 Frequency Tables: Discrete Ungrouped Data

In this section we revise collecting data to draw vertical line graphs or pie charts.

Example 1

The pupils in Mr Middleton's class take a maths test and get scores out of 10, which are listed below:

3 7 6 2 5 9 10 8 7 1
8 4 3 5 6 7 8 7 6 5
3 6 9 8 7 5 9 6 7 8

Illustrate these results using a pie chart.

Solution

First construct a tally chart and then calculate the angles for the pie chart.

<table>
<thead>
<tr>
<th>Score</th>
<th>Tally</th>
<th>Frequency</th>
<th>Angle</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>I</td>
<td>1</td>
<td>$\frac{360}{30} \times 1 = 12^\circ$</td>
</tr>
<tr>
<td>2</td>
<td>I</td>
<td>1</td>
<td>$\frac{360}{30} \times 1 = 12^\circ$</td>
</tr>
<tr>
<td>3</td>
<td>I I I</td>
<td>3</td>
<td>$\frac{360}{30} \times 3 = 36^\circ$</td>
</tr>
<tr>
<td>4</td>
<td>I</td>
<td>1</td>
<td>$\frac{360}{30} \times 1 = 12^\circ$</td>
</tr>
<tr>
<td>5</td>
<td>I I I I</td>
<td>4</td>
<td>$\frac{360}{30} \times 4 = 48^\circ$</td>
</tr>
<tr>
<td>6</td>
<td>I I I I I I</td>
<td>5</td>
<td>$\frac{360}{30} \times 5 = 60^\circ$</td>
</tr>
<tr>
<td>7</td>
<td>I I I I I I</td>
<td>6</td>
<td>$\frac{360}{30} \times 6 = 72^\circ$</td>
</tr>
<tr>
<td>8</td>
<td>I I I I I I</td>
<td>5</td>
<td>$\frac{360}{30} \times 5 = 60^\circ$</td>
</tr>
<tr>
<td>9</td>
<td>I I I</td>
<td>3</td>
<td>$\frac{360}{30} \times 3 = 36^\circ$</td>
</tr>
<tr>
<td>10</td>
<td>I</td>
<td>1</td>
<td>$\frac{360}{30} \times 1 = 12^\circ$</td>
</tr>
</tbody>
</table>

TOTAL 30 360 °
The pie chart can now be drawn as shown below:

Note: Remember to always give a title for the pie chart.

Example 2
Mrs Panni gave her class the same maths test as Mr Middleton's class in Example 1. The scores for her class are given below:

\[
\begin{array}{cccccccccccc}
3 & 4 & 4 & 4 & 5 & 6 & 7 & 7 & 4 & 3 \\
5 & 7 & 6 & 7 & 5 & 8 & 3 & 5 & 6 & 4 \\
6 & 7 & 7 & 6 & 6 & 4 & 6 & 5 & 6 & 7 \\
\end{array}
\]

Draw vertical line graphs for both classes and comment on the differences that they show.

Solution
First construct a tally chart for Mrs Panni's class.
The vertical line graph for Mrs Panni’s class is shown below:

<table>
<thead>
<tr>
<th>Score</th>
<th>Tally</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td></td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>I I I</td>
<td>3</td>
</tr>
<tr>
<td>4</td>
<td>I I I</td>
<td>6</td>
</tr>
<tr>
<td>5</td>
<td>I I I</td>
<td>5</td>
</tr>
<tr>
<td>6</td>
<td>I I I</td>
<td>8</td>
</tr>
<tr>
<td>7</td>
<td>I I I</td>
<td>7</td>
</tr>
<tr>
<td>8</td>
<td>I I I</td>
<td>1</td>
</tr>
<tr>
<td>9</td>
<td></td>
<td>0</td>
</tr>
<tr>
<td>10</td>
<td></td>
<td>0</td>
</tr>
</tbody>
</table>
The vertical line graph for Mr Middleton's class is shown below:

From the graphs we can see that Mr Middleton's class has some pupils who did better than those in Mrs Panni's class, but it also has some pupils who had lower scores.

Exercises

1. Emma stops at the park on her way to school every day for 2 weeks to look for conkers. The number of conkers she finds each day is listed below:

   2 10 1 2 3 4 5 3 2 4

   (a) Draw a pie chart to illustrate these data.

   (b) Can you think of a possible reason for the unusual number in the data?

2. Mr Rafiq runs a video library. Over a period of 3 days he notes how many videos have been borrowed during each hour. His records are shown below:

   3 2 4 3 8 7 9 3 10 2

   4 6 8 4 12 10 8 5 6 4

   3 2 6 9 4 5 7 6 5 7

   (a) Draw a pie chart to illustrate these data.

   (b) What was the largest number of videos hired in any hour?

   (c) What was the least number of videos hired in any hour?

   (d) What was the most common number of videos hired in any hour?
3. The total number of goals scored in each of the Premier League matches one Saturday were:

   0 1 0 4 5 4 2 3 1 4

   (a) Illustrate these data on a pie chart.
   (b) Which number of goals was the most common?

4. The pie chart below illustrates the scores obtained on a maths test by a class of 24 pupils:

   (a) How many degrees represent each pupil?
   (b) How many pupils get each score?

5. The National Curriculum levels reached in maths by a class at Key Stage 3 were:

   6 5 6 4 3 4 4 5 6 4 5 6 6 5
   5 6 4 4 6 6 5 5 6 4 6 5 4 5

   (a) Draw a vertical line diagram for this data.
   (b) Which level was obtained by most pupils?
6. The class in question 5 were also given National Curriculum levels for English and these are listed below:

\[
\begin{array}{cccccccccccc}
6 & 4 & 5 & 3 & 4 & 3 & 4 & 6 & 5 & 4 & 6 & 5 & 6 \\
4 & 6 & 4 & 5 & 5 & 5 & 4 & 5 & 5 & 3 & 6 & 4 & 4 \\
\end{array}
\]

(a) Draw a vertical line graph for the data.
(b) Compare this graph with the one for question 5 and comment on any differences.

7. A calculator was used to produce 40 random digits. These are listed below:

\[
\begin{array}{cccccccccccc}
7 & 2 & 1 & 0 & 5 & 4 & 6 & 8 & 1 & 2 \\
9 & 9 & 2 & 1 & 0 & 3 & 5 & 3 & 7 & 6 \\
5 & 4 & 1 & 0 & 9 & 3 & 9 & 9 & 4 & 5 \\
6 & 7 & 8 & 0 & 1 & 2 & 3 & 9 & 7 & 1 \\
\end{array}
\]

(a) Draw a vertical line graph to illustrate this data.
(b) Do you think that the calculator is good at producing random numbers? How does the graph support your answer?

8. A train company keeps a daily record of the number of trains that were late for two different months and these data are shown below:

\[
\begin{array}{cccccccccccc}
January & 1 & 0 & 2 & 7 & 3 & 1 & 2 & 4 & 5 & 1 & 2 \\
& 4 & 7 & 4 & 3 & 0 & 1 & 4 & 1 & 3 & 2 & 1 \\
& 2 & 3 & 4 & 5 & 6 & 0 & 1 & 2 & 9 \\
February & 5 & 6 & 5 & 4 & 3 & 7 & 8 & 1 & 4 & 3 & 4 \\
& 5 & 6 & 4 & 3 & 4 & 2 & 6 & 4 & 8 & 9 & 5 \\
& 3 & 4 & 3 & 5 & 4 & 4 \\
\end{array}
\]

(a) Draw a vertical line graph for each month.
(b) In which month was the better service provided to passengers?
(c) Why is the amount of data different for each of the two months?
(d) If January had the same number of days as February, would your answer to (b) be the same?
9. Peter grows tomatoes in his greenhouse. During August he keeps records of the number of tomatoes he picks each day. The results for 1997 and 1998 are shown below:

<table>
<thead>
<tr>
<th>Year</th>
<th>Tomatoes Picked</th>
</tr>
</thead>
<tbody>
<tr>
<td>1997</td>
<td>6 7 8 10 12 14 10 15 12 9 6</td>
</tr>
<tr>
<td></td>
<td>8 10 8 7 9 16 20 19 12 10 11</td>
</tr>
<tr>
<td></td>
<td>12 10 9 8 7 9 11 12 8</td>
</tr>
<tr>
<td>1998</td>
<td>11 11 14 16 15 12 12 11 8 9 20</td>
</tr>
<tr>
<td></td>
<td>19 17 18 19 15 16 17 15 12 11 10</td>
</tr>
<tr>
<td></td>
<td>8 17 18 20 15 16 17 19 15</td>
</tr>
</tbody>
</table>

(a) Draw vertical line graphs for each year.
(b) Describe how the two years compare.

5.2 Mean, Median, Mode and Range

The mean, median and mode are types of average. The range gives a measure of the spread of a set of data.

<table>
<thead>
<tr>
<th>Definition</th>
<th>Example</th>
</tr>
</thead>
</table>
| **Mean** = \( \frac{\text{sum of all data}}{\text{number of values}} \) | For 1, 2, 2, 3, 4  
  \[ Mean = \frac{1 + 2 + 2 + 3 + 4}{5} \]
  \[ = \frac{12}{5} \]
  \[ = 2.4 \] |
| **Mode** = most common value      | For 1, 2, 2, 3, 4  
  Mode = 2  
  For 1, 2, 2, 3, 4, 4, 5  
  Mode = 2 and 4 |
| **Median** = middle value when data is arranged in order | For 1, 2, 2, 3, 4  
  Median = 2  
  For 1, 2, 2, 3, 4, 4  
  Median = \( \frac{2 + 3}{2} \)  
  = 2.5 |
Definition Example

\[ \text{Range} = \text{largest value} - \text{smallest value} \]

For 1, 2, 2, 3, 4

\[ \text{Range} = 4 - 1 \]
\[ = 3 \]

The \textit{mean}, \textit{median}, \textit{mode} and \textit{range} can also be calculated when the data is presented in the form of a frequency table.

Example 1

For the data presented in the table opposite, calculate:

(a) the \textit{mode},
(b) the \textit{range},
(c) the \textit{median},
(d) the \textit{mean}.

\[
\begin{array}{|c|c|}
\hline
\text{Score} & \text{Frequency} \\
\hline
0 & 2 \\
1 & 6 \\
2 & 12 \\
3 & 4 \\
4 & 1 \\
\hline
\end{array}
\]

Solution

(a) The mode is the most common score. In this case,
\[ \text{mode} = 2 \]

(b) Largest score = 4, smallest score = 0,
\[ \text{range} = 4 - 0 \]
\[ = 4 \]

(c) The median is the middle value. As there are 25 scores, the middle value is the 13th score (12 above and 12 below).

When in order:
the first 2 values are 0,
the next 6 values are 1,
therefore the 3rd to 8th values are 1.

The next 12 values are 2,
therefore the 9th to 20th values are 2.
The 13th value is in this group,
so the 13th value is 2.

So the median = 2
(d) To calculate the mean, complete a table like the one below:

<table>
<thead>
<tr>
<th>Score</th>
<th>Frequency</th>
<th>Score × Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2</td>
<td>$0 \times 2 = 0$</td>
</tr>
<tr>
<td>1</td>
<td>6</td>
<td>$1 \times 6 = 6$</td>
</tr>
<tr>
<td>2</td>
<td>12</td>
<td>$2 \times 12 = 24$</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>$3 \times 4 = 12$</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>$4 \times 1 = 4$</td>
</tr>
</tbody>
</table>

**Totals**
- Score: 25
- Score × Frequency: 46

Mean = \( \frac{46}{25} \)
= 1.84

**Example 2**

Calculate the mean and median for the data in the table opposite.

**Solution**

(a) To calculate the mean, construct a table like the one below:

<table>
<thead>
<tr>
<th>Price (pence)</th>
<th>Frequency</th>
<th>Price × Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>30</td>
<td>1</td>
<td>$30 \times 1 = 30$</td>
</tr>
<tr>
<td>31</td>
<td>3</td>
<td>$31 \times 3 = 93$</td>
</tr>
<tr>
<td>32</td>
<td>4</td>
<td>$32 \times 4 = 128$</td>
</tr>
<tr>
<td>33</td>
<td>8</td>
<td>$33 \times 8 = 264$</td>
</tr>
<tr>
<td>34</td>
<td>2</td>
<td>$34 \times 2 = 68$</td>
</tr>
<tr>
<td>35</td>
<td>2</td>
<td>$35 \times 2 = 70$</td>
</tr>
</tbody>
</table>

**Totals**
- Price: 20
- Price × Frequency: 653

Mean = \( \frac{653}{20} \)
= 32.65p
As there are 20 values, the median will be between the 10th and 11th values.

When in order:
- 1st value is 30p
- Next 3 values are 31p
  - So the 2nd to 4th values are 31p.
- Next 4 values are 32p
  - So the 5th to 8th values are 32p
- Next 8 values are 33p
  - So 9th to 17th values are 33p.

So both the 10th and 11th values are 33p.

Median = 33p

Note: If the 10th and 11th values had been different from one another, we would have used a value halfway between them (the mean of the two numbers).

The symbol $\sum$ (Greek 'sigma') means 'the sum of' or 'the total of'.

So,

$$\text{mean} = \frac{\sum \text{frequency} \times \text{value}}{\sum \text{frequency}}$$

Sometimes we use $f$ to stand for frequency, $x$ for the values and $\bar{x}$ for the mean, so that,

$$\bar{x} = \frac{\sum fx}{\sum f}$$

Exercises

1. Calculate the mean, median, mode and range for each set of data below:
   (a) 3, 6, 3, 7, 4, 3, 9
   (b) 11, 10, 12, 12, 9, 10, 14, 12, 9
   (c) 6, 9, 10, 7, 8, 5
   (d) 2, 9, 7, 3, 5, 5, 6, 5, 4, 9.
2. (a) Copy and complete the table below:

<table>
<thead>
<tr>
<th>Score</th>
<th>Frequency</th>
<th>Score × Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>6</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>8</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>Totals</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(b) Calculate the mean score.

3. The number of goals scored by a hockey team in each match of a season is listed below:

6 2 2 3 1 0 0 1 2 3
3 5 4 2 0 1 1 0 2 1
1 1 1 2 1 1 0 2 3 4

(a) Copy and complete the table below:

<table>
<thead>
<tr>
<th>No. of Goals</th>
<th>Tally</th>
<th>Frequency</th>
<th>No. of Goals × Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(b) Calculate the mean.

(c) Calculate the median.

(d) What is the mode?

(e) What is the range?
4. The price of a litre of petrol at some garages was recorded and the results are given in the table opposite:

<table>
<thead>
<tr>
<th>Price</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>74p</td>
<td>1</td>
</tr>
<tr>
<td>75p</td>
<td>2</td>
</tr>
<tr>
<td>76p</td>
<td>8</td>
</tr>
<tr>
<td>77p</td>
<td>10</td>
</tr>
<tr>
<td>78p</td>
<td>2</td>
</tr>
<tr>
<td>79p</td>
<td>1</td>
</tr>
<tr>
<td>80p</td>
<td>1</td>
</tr>
</tbody>
</table>

Calculate the mean, median and mode of these data.

5. A class collected data on the number of children in their families, and this information is listed below:

2  1  3  4  2  5  3  1  2  1
1  1  2  3  2  2  2  3  4  2
1  1  1  2  3  2  1  1  2  3

(a) Calculate the mean number of children per family.
(b) Calculate the median number of children per family.
(c) Why are there no 'zeros'?

6. Professor Baker keeps a record of his golf scores, as shown in the table opposite:

<table>
<thead>
<tr>
<th>Golf Score</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>70</td>
<td>3</td>
</tr>
<tr>
<td>71</td>
<td>4</td>
</tr>
<tr>
<td>72</td>
<td>4</td>
</tr>
<tr>
<td>73</td>
<td>3</td>
</tr>
<tr>
<td>74</td>
<td>2</td>
</tr>
</tbody>
</table>

Calculate his mean score.

7. A class collected data on their shoe sizes and presented it in the table opposite:

<table>
<thead>
<tr>
<th>Shoe Size</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>4</td>
<td>7</td>
</tr>
<tr>
<td>5</td>
<td>6</td>
</tr>
<tr>
<td>6</td>
<td>5</td>
</tr>
<tr>
<td>7</td>
<td>3</td>
</tr>
<tr>
<td>8</td>
<td>2</td>
</tr>
</tbody>
</table>

(a) Calculate the mean, median and mode for the data.
(b) Which of the three types of average is the most useful to a shoe shop manager ordering stock?
8. Fataimenta sells vacuum cleaners and the table shows how many she sells each day in a 25-day period.

<table>
<thead>
<tr>
<th>No. Sold per Day</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>1</td>
<td>6</td>
</tr>
<tr>
<td>2</td>
<td>6</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>4</td>
<td>8</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
</tr>
</tbody>
</table>

(a) Calculate the mean, median and mode for the data.

(b) Which of the averages gives the best impression of her sales figures?

9. Classes 8A and 8B have a sponsored spelling competition. The tables below give the number of correct spellings for both classes.

<table>
<thead>
<tr>
<th>Class A</th>
<th>Class B</th>
</tr>
</thead>
<tbody>
<tr>
<td>No. of Correct Spellings</td>
<td>Frequency</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
</tr>
<tr>
<td>6</td>
<td>6</td>
</tr>
<tr>
<td>7</td>
<td>5</td>
</tr>
<tr>
<td>8</td>
<td>6</td>
</tr>
<tr>
<td>9</td>
<td>1</td>
</tr>
<tr>
<td>10</td>
<td>3</td>
</tr>
</tbody>
</table>

(a) Calculate the mean, median and mode for each class.

(b) Which average makes class A appear to be better at spelling?

(c) Which average makes class B appear to be better at spelling?

10. Paul and David play golf. The scores for their last 20 matches are given below:

<table>
<thead>
<tr>
<th>Score</th>
<th>67</th>
<th>68</th>
<th>69</th>
<th>70</th>
<th>71</th>
<th>72</th>
<th>73</th>
<th>74</th>
<th>75</th>
<th>76</th>
<th>77</th>
<th>78</th>
</tr>
</thead>
<tbody>
<tr>
<td>Paul's Frequency</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>5</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>David's Frequency</td>
<td>0</td>
<td>0</td>
<td>3</td>
<td>6</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>4</td>
<td>3</td>
</tr>
</tbody>
</table>

Produce arguments, supported by an average for each player, to show that each could be considered the better player.
## 6 Nets and Surface Area

### 6.1 Common 2-D and 3-D Shapes

You have already met many 2-D shapes; here are some with which you should already be familiar:

<table>
<thead>
<tr>
<th>NAME</th>
<th>ILLUSTRATION</th>
<th>NOTES</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Circle</strong></td>
<td></td>
<td>Symmetric about any diameter</td>
</tr>
<tr>
<td><strong>Triangle</strong></td>
<td></td>
<td>3 straight sides</td>
</tr>
<tr>
<td><strong>Equilateral Triangle</strong></td>
<td></td>
<td>3 equal sides and 3 equal angles (= 60 °)</td>
</tr>
<tr>
<td><strong>Isosceles Triangle</strong></td>
<td></td>
<td>2 equal sides and 2 equal angles</td>
</tr>
<tr>
<td><strong>Right-angled Triangle</strong></td>
<td></td>
<td>One angle = 90 °</td>
</tr>
<tr>
<td><strong>Quadrilateral</strong></td>
<td></td>
<td>4 straight sides</td>
</tr>
<tr>
<td><strong>Square</strong></td>
<td></td>
<td>4 equal sides and 4 right angles</td>
</tr>
<tr>
<td><strong>Rectangle</strong></td>
<td></td>
<td>Opposite sides equal and 4 right angles</td>
</tr>
<tr>
<td><strong>Rhombus</strong></td>
<td></td>
<td>4 equal sides; opposite sides parallel</td>
</tr>
<tr>
<td><strong>Trapezium</strong></td>
<td></td>
<td>One pair of opposite sides parallel</td>
</tr>
<tr>
<td><strong>Parallelogram</strong></td>
<td></td>
<td>Both pairs of opposite sides equal and parallel</td>
</tr>
<tr>
<td><strong>Kite</strong></td>
<td></td>
<td>Two pairs of adjacent sides equal</td>
</tr>
</tbody>
</table>
There are also several 3-D shapes with which you should be familiar:

<table>
<thead>
<tr>
<th>NAME</th>
<th>ILLUSTRATION</th>
<th>NOTES</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pentagon</td>
<td><img src="image" alt="Pentagon" /></td>
<td>5 sides (equal if regular)</td>
</tr>
<tr>
<td>Hexagon</td>
<td><img src="image" alt="Hexagon" /></td>
<td>6 sides (equal if regular)</td>
</tr>
<tr>
<td>Octagon</td>
<td><img src="image" alt="Octagon" /></td>
<td>8 sides (equal if regular)</td>
</tr>
<tr>
<td>Cube</td>
<td><img src="image" alt="Cube" /></td>
<td>All side lengths equal (square faces), and all angles right angles</td>
</tr>
<tr>
<td>Cuboid</td>
<td><img src="image" alt="Cuboid" /></td>
<td>Faces are combination of rectangles (and squares); all angles right angles</td>
</tr>
<tr>
<td>Cylinder</td>
<td><img src="image" alt="Cylinder" /></td>
<td>Circular base</td>
</tr>
<tr>
<td>Sphere</td>
<td><img src="image" alt="Sphere" /></td>
<td>All points on surface equidistant from centre</td>
</tr>
<tr>
<td>Pyramid</td>
<td><img src="image" alt="Pyramid" /></td>
<td>All slant edges are equal in length in a right pyramid</td>
</tr>
<tr>
<td>Prism</td>
<td><img src="image" alt="Prism" /></td>
<td>Cross-section remains the same throughout</td>
</tr>
<tr>
<td>Tetrahedron</td>
<td><img src="image" alt="Tetrahedron" /></td>
<td>All four faces are triangular</td>
</tr>
</tbody>
</table>

Note that a square is a special case of a rectangle, as it satisfies the definition; similarly, both a square and a rectangle are special cases of a parallelogram, etc.
Example 1
What is the name of the 2-D shape with 4 sides and with opposite angles equal?

Solution
The shape has to be a parallelogram.
(Note: this shape can also be a square, rhombus or rectangle as these are all special cases of a parallelogram.)

Example 2
Draw accurately:
(a) a rhombus with sides of length 4 cm and one angle $120^\circ$,
(b) a kite with sides of length 3 cm and 4 cm, and smallest angle $60^\circ$. Measure the size of each of the other angles.

Solution
(a)

(b) Note that the smallest angle, $60^\circ$, must be between the two longest sides. The other angles are approximately $108^\circ$, $108^\circ$ and $84^\circ$. 
Exercises

1. What could be the name of the 2-dimensional shape with 4 sides, which has all angles of equal sizes?

2. What is the name of a 6-sided, 2-dimensional shape which has sides of equal lengths?

3. Draw a parallelogram with sides of lengths 3 cm and 4 cm and with smallest angle equal to $60^\circ$.

4. Can a 4-sided, 2-dimensional shape have 4 sides of equal lengths, and not be a square?

5. Can a 4-sided, 2-dimensional shape have 4 angles of equal size, and not be a square?

6. Name all possible 4-sided, 2-dimensional shapes that have at least 2 sides of equal lengths.

7. Name all possible 4-sided, 2-dimensional shapes that have at most 2 sides of equal lengths.

6.2 2-D Representation of 3-D Shapes

In this section we explore how to draw 3-D shapes, either on squared paper or on isometric (triangular spotty) paper. Examples of each for a 2 cm cube, are shown below:

Example 1

On isometric paper, draw a cuboid with sides of lengths 5 cm, 3 cm and 2 cm.
Solution

The diagrams below show three of the possible ways of drawing a 2 cm × 3 cm × 5 cm cuboid.
Example 2

A triangular prism has a cross-section that is a right-angled triangle with base 4 cm and height 5 cm. The length of the prism is 8 cm.

Draw the prism.

Solution

First draw the cross-section of the prism. Then draw two lines of length 8 cm, parallel to each other. Complete the triangle at the other end of the prism.

Note: Lines parallel on the object are parallel on the diagram.
Example 3
Draw this prism on isometric paper:

Solution

Exercises

(Diagrams to be drawn full size unless scale given.)

1. On isometric paper, draw a cube with sides of length 4 cm.

2. On isometric paper, draw a cuboid with sides of lengths 3 cm, 2 cm and 4 cm.

3. Three cubes with sides of length 2 cm are put side-by-side to form a cuboid. Draw this cuboid on isometric paper.

4. A cuboid has sides of lengths 3 cm, 6 cm and 2 cm. Draw three possible views of the cuboid on isometric paper.

5. The cuboid shown in the diagram opposite may be cut in half to form two triangular prisms. Draw one of these prisms on isometric paper.

Note: The cut may be made in three different ways.
6. A triangular prism has a cross-section that is a right-angled triangle with base 4 cm and height 3 cm. The length of the prism is 6 cm. Draw the prism on isometric paper.

7. On plain or squared paper, draw a cube with sides of 5 cm.

8. On plain or squared paper, draw a cuboid with sides of lengths 6 cm, 4 cm and 3 cm.

9. A prism has a triangular cross-section with sides of length 6 cm. The length of the prism is 8 cm. Draw the prism on plain paper.

10. The diagram shows the cross-section of a triangular prism. The length of the prism is 5 cm. Draw the prism on plain paper.

6.3 Plans and Elevations

The plan of a solid is the view looking \textit{down from above}.

\textit{Side} and \textit{front elevations} are drawn as if looking at the solid from the \textit{side} or the \textit{front}, where the front is taken to be the face nearest to you.

Example 1

Draw the \textit{plan} and \textit{elevations} of this cuboid:
Solution

The plan is the view from above:

The front elevation is the view from the front:

The side elevation is the view from the side (in this case the right and left side elevations are the same):

Example 2

Draw the plan, front elevation and left side elevation for this shed:

Solution

Using 1 cm for 1 m:
Exercises

*(Diagrams to be drawn full size unless scale given.)*

1. Draw the plan and elevations of the cuboid shown:

2. Draw the plan and elevations of the triangular prism shown:

3. Draw the plan and elevations of the building shown, which is 4 m high:
   Use a scale of 1 cm to represent 1 m.
4. (a) Draw the plan and elevations of the building shown using a scale of 1 cm for 1 m:
(b) How do these views compare with those in Example 2 and in question 3?

5. A square-based right pyramid has a base with sides of length 4 cm. The sides of the pyramid are isosceles triangles, and the vertical height of the pyramid is 5 cm. Draw the plan, and an elevation of the pyramid.

6. The diagram shows a tissue box. The opening in the centre of the top of the box is 8 cm by 4 cm.

7. A hole of radius 1 cm is drilled through the middle of a block of wood as shown in the diagram:

Draw the plan and elevations of the block of wood.
8. Draw the plan and elevations of the barn shown opposite:
Use a scale of 1 cm for 1 m.

9. The sketch shows the design of a house with an overhanging roof.

Draw the plan and elevations of the house.

10. The diagram shows a factory with a flat roof and a square-based chimney:
Draw the plan and elevations of the building, using a scale of 1 cm for 1 m.
6.4 Nets and Surface Area of Cubes and Cuboids

A net can be folded up to make a solid. The diagram below shows one of the possible nets of a cube:

The net of a cube is always made up of 6 squares. Each square has an area of $x^2$ if the length of the side of the cube is $x$.

Total surface area of a cube $= 6x^2$.

Example 1

Draw a net for the cube shown and calculate its surface area.
Solution

The net is made up of 6 squares.
Each square has an area of $4 \text{ cm}^2$.
Surface area $= 6 \times 4$
$= 24 \text{ cm}^2$.

The net of a cuboid is made up of 6 rectangles.
The rectangles will occur in pairs as illustrated below:

- Top and bottom
- Two sides
- Two ends

For this cuboid,

and, surface area $= xy + yz + xz + xy + yz + xz$
$= 2xy + 2yz + 2xz$
$= 2(xy + yz + xz)$
Example 2

Draw a net for the cuboid shown and calculate its surface area.

Solution

One of the possible nets for the cuboid is shown opposite, together with the area of each rectangle:

\[
\text{Surface area} = 2 + 6 + 3 + 6 + 3 + 2 \\
= 22 \text{ cm}^2
\]

You can check your solution:

\[x = 2 \text{ cm}, \ y = 3 \text{ cm} \text{ and } z = 1 \text{ cm}\]

so, using the formula \(2(xy + yz + xz)\),

\[
\text{surface area} = 2(2 \times 3 + 3 \times 1 + 2 \times 1) \\
= 2 \times 11 \\
= 22 \text{ cm}^2 \text{ (as before)}
\]

Example 3

Calculate the surface area of this cuboid:

Solution

\[
\text{Surface area} = 2(5 \times 1 + 1 \times 8 + 5 \times 8) \\
= 2(5 + 8 + 40) \\
= 2 \times 53 \\
= 106 \text{ cm}^2
\]
Exercises

1. Draw different arrangements of 6 squares and indicate which of them could be folded to form a cube.

2. Draw a net for a cube with sides of length 4 cm, and calculate its surface area.

3. Draw a net for the cuboid shown, and calculate its surface area.

4. (a) On card, draw a net for a cube with sides of length 5 cm.
   (b) Add tabs to the net so that it can be cut out and glued together.
   (c) Cut out the net, fold it up and glue it together to make a cube.

5. Use card to make a net for the cuboid shown. Then add tabs, cut it out, fold it up and glue it to make the cuboid.

6. (a) Draw 2 different nets for the cuboid shown.
   (b) Calculate the surface area of the cuboid.
   (c) Do both your nets have the same surface areas?

7. Without drawing a net, calculate the surface area of a cube with sides of length:
   (a) 10 cm       (b) 9 cm.
8. Calculate the surface area of each of the following cuboids:

(a) 

(b) 

(c) 

(d) 

9. A diagram of a net is shown below, where two of the rectangles have been drawn inaccurately.

(a) Explain what is wrong with the net.
(b) Draw a modified net that would produce a cuboid, by changing two of the rectangles.
(c) Give an alternative answer to part (b).

10. The surface area of a cube is 24 cm\(^2\). Calculate the length of the sides of the cube.

11. The surface area of this cuboid is 102 cm\(^2\). What is the length marked \(x\) ?
6.5 Nets of Prisms and Pyramids

In order to draw the nets of some prisms and pyramids, you will need to construct triangles as well as squares and rectangles.

Example 1
(a) Draw a net for this triangular prism:
(b) Calculate its surface area.

Solution
(a) A net is shown below where all lengths marked are in cm.

(b) The area of each part of the net has been calculated.

\[
\text{Surface area} = (5 \times 4) + (4 \times 4) + (4 \times 3) + \left(\frac{1}{2} \times 4 \times 3\right) + \left(\frac{1}{2} \times 4 \times 3\right) \\
= 20 + 16 + 12 + 6 + 6 \\
= 60 \text{ cm}^2
\]
Example 2

The square base of a pyramid has sides of length 4 cm. The triangular faces of the pyramid are all isosceles triangles with two sides of length 5 cm.

Draw a net for the pyramid.

Solution

Exercises

1. Draw a net for the triangular prism shown opposite:

2. Draw a net for this prism, on card. Add tabs, cut it out, and then construct the actual prism.
3. A pyramid has a square base with sides of length 6 cm. The other edges of the prism have length 6 cm. Draw a net for the pyramid.

4. A pyramid has a rectangular base with sides of lengths 3 cm and 4 cm. The other edges of the pyramid have length 6 cm.
   Draw a net for this pyramid on card, cut it out and construct the pyramid.

5. A tetrahedron has four faces which are all equilateral triangles. Draw a net for a tetrahedron, which has edges of length 4 cm.

6. A square-based prism has a base with sides of length 5 cm and vertical height 6 cm. Draw the net of this prism.

7. The diagram shows a prism:
   (a) Draw a net for the prism.
   (b) Find the height of the prism.

8. A container is in the shape of a pyramid on top of a cuboid, as shown in the diagram opposite.
   Draw a net for the container.

9. The diagram below shows a square-based pyramid; the base is horizontal and AE is vertical. Draw a net for this pyramid.
7 Ratio and Proportion

7.1 Equivalent Ratios

Orange squash is to be mixed with water in a ratio of 1 : 6; this means that for every unit of orange squash, 6 units of water will be used. The table gives some examples:

<table>
<thead>
<tr>
<th>Amount of Orange Squash (cm³)</th>
<th>Amount of Water (cm³)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>6</td>
</tr>
<tr>
<td>20</td>
<td>120</td>
</tr>
<tr>
<td>5</td>
<td>30</td>
</tr>
</tbody>
</table>

The ratios 1 : 6 and 20 : 120 and 5 : 30 are all equivalent ratios, but 1 : 6 is the simplest form.

Ratios can be simplified by dividing both sides by the same number: note the similarity to fractions. An alternative method for some purposes, is to reduce to the form 1 : n or n : 1 by dividing both numbers by either the left-hand-side (LHS) or the right-hand-side (RHS). For example:

the ratio 4 : 10 may be simplified to \(\frac{4}{4} : \frac{10}{4} \Rightarrow 1 : 2.5\)

the ratio 8 : 5 may be simplified to \(\frac{8}{5} : \frac{5}{5} \Rightarrow 1.6 : 1\)

Example 1

Write each of these ratios in its simplest form:

(a) 7 : 14  
(b) 15 : 25  
(c) 10 : 4

Solution

(a) Divide both sides by 7, giving

\[7 : 14 = \frac{7}{7} : \frac{14}{7} = 1 : 2\]

(b) Divide both sides by 5, giving

\[15 : 25 = \frac{15}{5} : \frac{25}{5} = 3 : 5\]

(c) Divide both sides by 2, giving

\[10 : 4 = \frac{10}{2} : \frac{4}{2} = 5 : 2\]
Example 2
Write these ratios in the form $1 : n$.
(a) $3 : 12$  
(b) $5 : 6$  
(c) $10 : 42$

Solution
(a) Divide both sides by 3, giving
$$3 : 12 = 1 : 4$$
(b) Divide both sides by 5, giving
$$5 : 6 = 1 : \frac{6}{5} = 1 : 1.2$$
(c) Divide both sides by 10, giving
$$10 : 42 = 1 : \frac{42}{10} = 1 : 4.2$$

Example 3
The scale on a map is $1 : 20000$. What actual distance does a length of 8 cm on the map represent?

Solution
Actual distance $= 8 \times 20000$
$= 160000$ cm
$= 1600$ m
$= 1.6$ km

Exercises
1. Write each of these ratios in its simplest form:
   (a) $2 : 6$  
   (b) $4 : 20$  
   (c) $3 : 15$  
   (d) $6 : 2$  
   (e) $24 : 4$  
   (f) $30 : 25$  
   (g) $14 : 21$  
   (h) $15 : 60$  
   (i) $20 : 100$  
   (j) $80 : 100$  
   (k) $18 : 24$  
   (l) $22 : 77$

2. Write in the form $1 : n$, each of the following ratios:
   (a) $2 : 5$  
   (b) $5 : 3$  
   (c) $10 : 35$  
   (d) $2 : 17$  
   (e) $4 : 10$  
   (f) $8 : 20$  
   (g) $6 : 9$  
   (h) $15 : 12$  
   (i) $5 : 12$
3. Write in the form \( n : 1 \), each of the following ratios:
   (a) \( 24 : 3 \)  \( \quad \) (b) \( 4 : 5 \)  \( \quad \) (c) \( 7 : 10 \)
   (d) \( 15 : 2 \)  \( \quad \) (e) \( 18 : 5 \)  \( \quad \) (f) \( 6 : 5 \)

4. Jennifer mixes 600 ml of orange juice with 900 ml of apple juice to make a fruit drink. Write the ratio of orange juice to apple juice in its simplest form.

5. A builder mixes 10 shovels of cement with 25 shovels of sand. Write the ratio of cement to sand:
   (a) in its simplest form,
   (b) in the form \( 1 : n \),
   (c) in the form \( n : 1 \),

6. In a cake recipe, 300 grams of butter are mixed with 800 grams of flour. Write the ratio of butter to flour:
   (a) in its simplest form,
   (b) in the form \( 1 : n \),
   (c) in the form \( n : 1 \).

7. In a school there are 850 pupils and 40 teachers. Write the ratio of teachers to pupils:
   (a) in its simplest form,  \( \quad \) (b) in the form \( 1 : n \).

8. A map is drawn with a scale of \( 1 : 50000 \). Calculate the actual distances, in km, that the following lengths on the map represent:
   (a) \( 2 \) cm  \( \quad \) (b) \( 9 \) cm  \( \quad \) (c) \( 30 \) cm.

9. A map has a scale of \( 1 : 200000 \). The distance between two towns is 60 km. How far apart are the towns on the map?

10. On a map, a distance of 5 cm represents an actual distance of 15 km. Write the scale of the map in the form \( 1 : n \).

7.2 Direct Proportion

Direct proportion can be used to carry out calculations like the one below:

- If 10 calculators cost £120,
- then 1 calculator costs £12,
- and 8 calculators cost £96.
Example 1
If 6 copies of a book cost £9, calculate the cost of 8 books.

Solution
If 6 copies cost £9,
then 1 copy costs $\frac{9}{6} = £1.50$
and 8 copies cost $£1.50 \times 8 = £12$

Example 2
If 25 floppy discs cost £5.50, calculate the cost of 11 floppy discs.

Solution
If 25 discs cost £5.50 = 550p
then 1 disc costs $\frac{550}{25} = 22p$
so 11 discs cost $11 \times 22p = 242p$
= £2.42

Exercises
1. If 5 tickets for a play cost £40, calculate the cost of:
   (a) 6 tickets (b) 9 tickets (c) 20 tickets.

2. To make 3 glasses of orange squash you need 600 ml of water. How much water do you need to make:
   (a) 5 glasses of orange squash,
   (b) 7 glasses of orange squash?

3. If 10 litres of petrol cost £8.20, calculate the cost of:
   (a) 4 litres (b) 12 litres (c) 30 litres.

4. A baker uses 1800 grams of flour to make 3 loaves of bread. How much flour will he need to make:
   (a) 2 loaves (b) 7 loaves (c) 24 loaves?
5. Ben buys 21 football stickers for 84p. Calculate the cost of:
   (a) 7 stickers  (b) 12 stickers  (c) 50 stickers.

6. A 20 m length of rope costs £14.40.
   (a) Calculate the cost of 12 m of rope.
   (b) What is the cost of the rope, per metre?

7. A window cleaner charges $n$ pence to clean each window, and for a house with 9 windows he charges £4.95.
   (a) What is $n$?
   (b) Calculate the window cleaner's charge for a house with 13 windows.

8. 16 teams, each with the same number of people, enter a quiz. At the semifinal stage there are 12 people left in the competition. How many people entered the quiz?

9. Three identical coaches can carry a total of 162 passengers. How many passengers in total can be carried on seven of these coaches?

10. The total mass of 200 concrete blocks is 1460 kg. Calculate the mass of 900 concrete blocks.

7.3 Proportional Division

Sometimes we need to divide something in a given ratio. Malcolm and Alison share the profits from their business in the ratio 2 : 3. This means that, out of every £5 profit, Malcolm gets £2 and Alison gets £3.

Example 1

Julie and Jack run a stall at a car boot sale and take a total of £90. They share the money in the ratio 4 : 5. How much money does each receive?

Solution

As the ratio is 4 : 5, first add these numbers together to see by how many parts the £90 is to be divided.

$4 + 5 = 9$, so 9 parts are needed.

Now divide the total by 9.

$\frac{90}{9} = 10$, so each part is £10.
Julie gets 4 parts at £10, giving $4 \times £10 = £40$.

Jack gets 5 parts at £10, giving $5 \times £10 = £50$.

**Example 2**

Rachel, Ben and Emma are given £52. They decide to divide the money in the ratio of their ages, 10 : 9 : 7. How much does each receive?

**Solution**

$10 + 9 + 7 = 26$ so 26 parts are needed.

Now divide the total by 26.

$$\frac{52}{26} = 2,$$

so each part is £2.

Rachel gets 10 parts at £2, giving $10 \times £2 = £20$

Ben gets 9 parts at £2, giving $9 \times £2 = £18$

Emma gets 7 parts at £2, giving $7 \times £2 = £14$

**Exercises**

1. (a) Divide £50 in the ratio 2 : 3.
   (b) Divide £100 in the ratio 1 : 4.
   (c) Divide £60 in the ratio 11 : 4.
   (d) Divide 80 kg in the ratio 1 : 3.

2. (a) Divide £60 in the ratio 6 : 5 : 1.
   (b) Divide £108 in the ratio 3 : 4 : 5.
   (c) Divide 30 kg in the ratio 1 : 2 : 3.
   (d) Divide 75 litres in the ratio 12 : 8 : 5.

3. Heidi and Briony get £80 by selling their old toys at a car boot sale. They divide the money in the ratio 2 : 3. How much money do they each receive?

4. In a chemistry lab, acid and water are mixed in the ratio 1 : 5. A bottle contains 216 ml of the mixture. How much acid and how much water were needed to make this amount of the mixture?

5. Blue and yellow paints are mixed in the ratio 3 : 5 to produce green. How much of each of the two colours are needed to produce 40 ml of green paint?
6. Simon, Sarah and Matthew are given a total of £300. They share it in the ratio 10 : 11 : 9. How much does each receive?

7. In a fruit cocktail drink, pineapple juice, orange juice and apple juice are mixed in the ratio 7 : 5 : 4. How much of each type of juice is needed to make:
   (a) 80 ml of the cocktail,
   (b) 1 litre of the cocktail?

8. Blue, red and yellow paints are mixed to produce 200 ml of another colour. How much of each colour is needed if they are mixed in the ratio:
   (a) 1 : 1 : 2,
   (b) 3 : 3 : 2,
   (c) 9 : 4 : 3?

9. To start up a small business, it is necessary to spend £800. Paul, Margaret and Denise agree to contribute in the ratio 8 : 1 : 7. How much does each need to spend?

10. Hannah, Grace and Jordan share out 10 biscuits so that Hannah has 2, Grace has 6 and Jordan has the remainder. Later they share out 25 biscuits in the same ratio. How many does each have this time?

### 7.4 Linear Conversion

The ideas used in this unit can be used for converting masses, lengths and currencies.

**Example 1**

If £1 is worth 9 French francs, convert:

(a) £22 to Ff,
(b) 45 Ff to £,
(c) 100 Ff to £.

**Solution**

(a) \[ \text{£22} = 22 \times 9 \]
\[ = 198 \text{ Ff} \]

(b) \[ 1 \text{ Ff} = \frac{1}{9} \text{ £} \]
\[ \text{so } 45 \text{ Ff} = 45 \times \frac{1}{9} \]
\[ = \frac{45}{9} \]
\[ = £5 \]
Example 2

Use the fact that 1 foot is approximately 30 cm to convert:

(a) 8 feet to cm,
(b) 50 cm to feet,
(c) 195 cm to feet.

Solution

(a) 8 feet = $8 \times 30$
    $= 240$ cm

(b) 1 cm = $\frac{1}{30}$ feet

so 50 cm = $50 \times \frac{1}{30}$
    $= \frac{5}{3}$
    $= 1 \frac{2}{3}$ feet

(c) 195 cm = $195 \times \frac{1}{30}$
    $= \frac{195}{30}$
    $= \frac{13}{2}$
    $= 6 \frac{1}{2}$ feet

Example 3

If £1 is worth $1.60, convert:

(a) £15 to dollars
(b) $8 to pounds.
Solution

(a) £15 = 15 × 1.60
    = $24

(b) $1 = \frac{\text{1}}{1.60}
    = \frac{\text{10}}{16}

$8 = 8 \times \frac{\text{10}}{16}
    = \frac{80}{16}
    = £5

Exercises

1. If £1 is worth 9 Ff, convert:
   (a) £6 to Fr,
   (b) £100 to Ff,
   (c) 54 Ff to £,
   (d) 28 Ff to £.

2. Use the fact that 1 inch is approximately 25 mm to convert:
   (a) 6 inches to mm,
   (b) 80 inches to mm,
   (c) 50 mm to inches,
   (d) 1000 mm to inches.

3. Before 1971, Britain used a system of money where there were 12 pennies in a shilling and 20 shillings in a pound. Use this information to convert:
   (a) 100 shillings into pounds,
   (b) 8 shillings into pennies,
   (c) 132 pennies into shillings,
   (d) 180 pennies into shillings.

4. Given that a weight of 1 lb is approximately equivalent to 450 grams, convert:
   (a) 5 lbs to grams,
   (b) 9 lb into grams,
   (c) 1800 grams to lb,
   (d) 3150 grams to lb.

5. Use the fact that 1 mile is approximately the same distance as 1.6 km to convert:
   (a) 30 miles to km,
   (b) 21 miles to km,
   (c) 80 km to miles,
   (d) 200 km to miles.
6. On a certain day, the exchange rate was such that £1 was worth $1.63. Use a calculator to convert the following amounts to £, giving each answer correct to the nearest pence.
   (a) $100  
   (b) $250  
   (c) $75.

7. The Japanese currency is the Yen (Y). The exchange rate gives 197 Yen for every £1. Using a calculator, convert the following amounts to pounds, giving each answer correct to the nearest pence.
   (a) 1000 Y  
   (b) 200 Y  
   (c) 50,000 Y.

8. A weight of 1 lb is approximately equivalent to 450 grams. There are 16 ounces in 1 lb. Give answers to the following questions correct to 1 decimal place.
   (a) Convert 14 oz to lb.
   (b) Convert 200 grams to lb.
   (c) Convert 300 grams to ounces.

9. If £1 is worth 2.8 German Marks (DM), and 1 DM is worth 2800 Italian Lira (L), use a calculator to convert:
   (a) 800 DM to £,  
   (b) 10,000 L to DM,  
   (c) 50,000 L to £.

10. There are 8 pints in one gallon. One gallon is equivalent to approximately 4.55 litres. Use a calculator to convert:
    (a) 12 pints to litres,  
    (b) 20 litres to pints.
    Give your answers correct to 1 decimal place.

7.5 Inverse Proportion

Inverse proportion is when an increase in one quantity causes a decrease in another.

The relationship between speed and time is an example of inverse proportionality: as the speed increases, the journey time decreases, so the time for a journey can be found by dividing the distance by the speed.

Example 1

(a) Ben rides his bike at a speed of 10 mph. How long does it take him to cycle 40 miles?

(b) On another day he cycles the same route at a speed of 16 mph. How much time does this journey take?
Solution

(a) Time = \( \frac{40}{10} \) = 4 hours

(b) Time = \( \frac{40}{16} \) = 2 \( \frac{1}{2} \) hours

Note: Faster speed \( \Rightarrow \) shorter time.

Example 2

Jai has to travel 280 miles. How long does it take if he travels at:

(a) 50 mph,
(b) 60 mph?
(c) How much time does he save when he travels at the faster speed?

Solution

(a) Time = \( \frac{280}{50} \) = 5.6 hours = 5 hours 36 minutes

(b) Time = \( \frac{280}{60} \) = 4 \( \frac{2}{3} \) hours = 4 hours 40 minutes

(c) Time saved = 5 hours 36 mins – 4 hours 40 mins = 56 minutes

Example 3

In a factory, each employee can make 40 chicken pies in one hour. How long will it take:

(a) 6 people to make 40 pies,
(b) 3 people to make 240 pies,
(c) 10 people to make 600 pies?
Solution

(a)  1 person makes 40 pies in 1 hour.
6 people make 40 pies in $\frac{1}{6}$ hour (or 10 minutes).

(b)  1 person makes 40 pies in 1 hour.
1 person makes 240 pies in $\frac{240}{40} = 6$ hours.
3 people make 240 pies in $\frac{6}{3} = 2$ hours.

(a)  1 person makes 40 pies in 1 hour.
1 person makes 600 pies in $\frac{600}{40} = 15$ hours.
10 people make 600 pies in $\frac{15}{10} = 1 \frac{1}{2}$ hours.

Exercises

1. How long does it take to complete a journey of 300 miles travelling at:
   (a) 60 mph,   (b) 50 mph,   (c) 40 mph?

2. Alec has to travel 420 miles. How much time does he save if he travels at 70 mph rather than 50 mph?.

3. Sarah has to travel 60 miles to see her boyfriend. Her dad drives at 30 mph and her uncle drives at 40 mph. How much time does she save if she travels with her uncle rather than with her dad?

4. Tony usually walks to school at 3 mph. When Jennifer walks with him he walks at 4 mph. He walks 1 mile to school. How much quicker is his journey when he walks with Jennifer?

5. One person can put 200 letters into envelopes in 1 hour. How long would it take for 200 letters to be put into envelopes by:
   (a) 4 people,
   (b) 6 people,
   (c) 10 people?
6. A person can make 20 badges in one hour using a machine. How long would it take:
   (a) 4 people with machines to make 20 badges,
   (b) 10 people with machines to make 300 badges,
   (c) 12 people with machines to make 400 badges?

7. A train normally complete a 270-mile journey in $4\frac{1}{2}$ hours. How much faster would it have to travel to complete the journey in 4 hours?

8. On Monday Tom takes 15 minutes to walk one mile to school. On Tuesday he takes 20 minutes to walk the same distance. Calculate his speed in mph for each day's walk.

   (a) How many kg of sweets do they each receive?
   (b) How much less would they each have received if there were four friends instead of three?

10. Nadina and her friends can each make 15 Christmas cards in one hour. How long would it take Nadina and four friends to make:
    (a) 300 cards,
    (b) 1000 cards?
# 8 Algebra: Brackets

## 8.1 Expansion of Single Brackets

In this section we consider how to expand (multiply out) brackets to give two or more terms, as shown below:

\[3(x + 6) = 3x + 18\]

First we revise *negative numbers* and *order of operations*.

### Example 1

Evaluate:

(a) \(-6 + 10\)
(b) \(-7 + (-4)\)
(c) \((-6) \times (-5)\)
(d) \(6 \times (4 - 7)\)
(e) \(4(8 + 3)\)
(f) \(6(8 - 15)\)
(g) \(3 - (-5)\)
(h) \(\frac{(-2) - (-3)}{-1}\)

### Solution

(a) \(-6 + 10 = 4\)

(b) \(-7 + (-4) = -7 - 4\)
\[= -11\]

(c) \((-6) \times (-5) = 30\)

(d) \(6 \times (4 - 7) = 6 \times (-3)\)
\[= -18\]

(e) \(4(8 + 3) = 4 \times 11\)
\[= 44\]

(f) \(6(8 - 15) = 6 \times (-7)\)
\[= -42\]
(g) \[ 3 - (-5) = 3 + 5 \]
\[ = 8 \]
(h) \[ \frac{(-2) - (-3)}{-1} = \frac{(-2) + 3}{-1} \]
\[ = \frac{1}{-1} \]
\[ = -1 \]

When a bracket is expanded, *every term* inside the bracket must be multiplied by the number outside the bracket. Remember to think about whether each number is positive or negative!

**Example 2**

Expand \(3(x + 6)\) using a table.

**Solution**

<table>
<thead>
<tr>
<th></th>
<th>x</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>3x</td>
<td>18</td>
</tr>
</tbody>
</table>

From the table,

\[ 3(x + 6) = 3x + 18 \]

**Example 3**

Expand \(4(x - 7)\).

**Solution**

\[ 4(x - 7) = 4x - 4 \times 7 \]
\[ = 4x - 28 \]

Remember that *every term* inside the bracket must be multiplied by the number outside the bracket.

**Example 4**

Expand \(x(8 - x)\).
Solution

\[ x (8 - x) = x \times 8 - x \times x \]
\[ = 8x - x^2 \]

Example 5

Expand \((-3) (4 - 2x)\).

Solution

\[ (-3)(4 - 2x) = (-3) \times 4 - (-3) \times 2x \]
\[ = -12 - (-6x) \]
\[ = -12 + 6x \]

Exercises

1. Calculate:
   (a) \(-6 + 17\)  
   (b) \(6 - 14\)  
   (c) \(-6 - 5\)  
   (d) \(6 - (-9)\)  
   (e) \(-11 - (-4)\)  
   (f) \((-6) \times (-4)\)  
   (g) \(8 \times (-7)\)  
   (h) \(88 \div (-4)\)  
   (i) \(6 (8 - 10)\)  
   (j) \(5 (3 - 10)\)  
   (k) \(7 (11 - 4)\)  
   (l) \((-4) (6 - 17)\)

2. Copy and complete the following tables, and write down each of the expansions:
   (a) \[
   \begin{array}{c|c|c}
   \times & x & 2 \\
   \hline
   4 & & \\
   \end{array}
   \]
   \[4 (x + 2) = \]
   (b) \[
   \begin{array}{c|c|c}
   \times & x & -7 \\
   \hline
   5 & & \\
   \end{array}
   \]
   \[5 (x - 7) = \]
   (c) \[
   \begin{array}{c|c|c}
   \times & x & 3 \\
   \hline
   4 & & \\
   \end{array}
   \]
   \[4 (x + 3) = \]
   (d) \[
   \begin{array}{c|c|c}
   \times & 2x & 5 \\
   \hline
   5 & & \\
   \end{array}
   \]
   \[5 (2x + 5) = \]
3. Expand:
   (a) $4(x + 6)$
   (b) $3(x - 4)$
   (c) $5(2x + 6)$
   (d) $7(3x - 4)$
   (e) $3(2x + 4)$
   (f) $8(3x - 9)$
   (g) $(-2)(x - 4)$
   (h) $(-3)(8 - 2x)$
   (i) $5(3x - 4)$
   (j) $9(2x + 8)$

   Explain why his expansion is not correct.

5. Copy and complete the following tables and write down each of the expansions:
   (a) $\times \quad x \quad -2$
      
      $\begin{array}{|c|c|}
      \hline
      x & \hline
      \end{array}$
   
   $x(x - 2) =$
   
   (b) $\times \quad x \quad -y$
      
      $\begin{array}{|c|c|}
      \hline
      x & \hline
      \end{array}$
   
   $x(x - y) =$

6. Copy the following expansions, filling in the missing terms:
   (a) $4(x + 8) = 4x^2 + ?$
   (b) $(-3)(2x - 7) = ? + 21$
   (c) $4x(x - 9) = 4x^2 - ?$
   (d) $6x(x - 7) = 6x^2 - ?$
   (e) $3x(x - y) = 3x^2 - ?$
   (f) $(-4x)(2x + 8) = ? - 32x$

7. Expand:
   (a) $x(x - 7)$
   (b) $x(8 - 2x)$
   (c) $6x(x + 2)$
   (d) $4x(3x - 5)$
   (e) $x(x + y)$
   (f) $x(4y - 3x)$
   (g) $2x(2x + 3y)$
   (h) $5x(2y - 1)$

8. Write down expressions for the area of each of these rectangles, and then expand the brackets:
   (a) $\begin{array}{|c|}
   \hline
   2
   \hline
   \end{array} \quad x + 4$
   (b) $\begin{array}{|c|}
   \hline
   12
   \hline
   \end{array} \quad x - 5$
9. Write down an expression for the area of this triangle, that:
(a) contains brackets,
(b) does not contain brackets.

10 Write down an expression for the volume of this cuboid, that:
(a) contains brackets,
(b) does not contain brackets.

8.2 Linear Equations
Expanding a bracket will usually be the first step when solving an equation like

\[ 4(x + 3) = 20 \]

Example 1
Solve

\[ 5(x - 3) = 35 \]

Solution

\[ 5(x - 3) = 35 \]
Expanding brackets gives:
\[ 5x - 15 = 35 \]
Adding 15 to both sides gives:
\[ 5x = 50 \]
Dividing by 5 gives:
\[ x = 10 \]
Example 2

Solve

\[ 6(x + 7) = 50 \]

Solution

\[ 6(x + 7) = 50 \]

Expanding brackets gives:

\[ 6x + 42 = 50 \]

Subtracting 42 from both sides gives:

\[ 6x = 8 \]

Dividing by 6 gives:

\[ x = \frac{8}{6} = \frac{4}{3} \]

Example 3

Gilda thinks of a number and adds 7 to it. She then multiplies her answer by 4 and gets 64.

(a) Write down an equation that can be used to calculate the number with which Gilda started.

(b) Solve your equation to give the number.

Solution

(a) Start with \( x \).

Add 7 to give \( x + 7 \)

Multiply by 4 to give \( 4(x + 7) \)

This expression equals 64, so the equation is \( 4(x + 7) = 64 \)

(b) \[ 4(x + 7) = 64 \]

Expanding brackets gives:

\[ 4x + 28 = 64 \]

Subtracting 28 from both sides gives:

\[ 4x = 36 \]

Dividing by 4 gives:

\[ x = \frac{36}{4} = 9 \]
Exercises

1. Solve these equations:
   (a) \(2(x + 6) = 14\)  
   (b) \(5(x - 8) = 40\)  
   (c) \(3(x + 5) = 12\)  
   (d) \(7(x + 4) = 42\)  
   (e) \(2(x + 7) = 19\)  
   (f) \(3(x - 4) = 11\)  
   (g) \(5(x - 4) = 12\)  
   (h) \(10(x + 7) = 82\)

2. Solve these equations:
   (a) \(5(2x - 7) = 8\)  
   (b) \(3(3x + 6) = 27\)  
   (c) \(3(2x + 1) = 30\)  
   (d) \(8(2x - 12) = 24\)

3. A rectangle has sides of length 3 m and \((x + 4)\) m. 
   Find the value of \(x\), if the area of the rectangle is 18 m\(^2\).

4. Feti chooses a number, adds 7, multiplies the result by 5 and gets the answer 55. 
   (a) If \(x\) is the number Feti first chose, write down an equation that can be used to determine the number.
   (b) Solve the equation to determine the value of \(x\).

5. The following flow chart is used to form an equation:

\[
x \rightarrow + 6 \rightarrow \times 4 \rightarrow 17
\]

(a) Write down the equation.
(b) Solve the equation to find the value of \(x\).

6. Solve the following equations:
   (a) \(4(7 - x) = 20\)  
   (b) \(3(9 - x) = 15\)  
   (c) \(6(5 - 2x) = 18\)  
   (d) \(5(7 - 3x) = 20\)  
   (e) \(2(10 - 3x) = 17\)  
   (f) \(6(9 - 5x) = 4\)
7. Alice thinks of a number, subtracts it from 11 and then multiplies her answer by 5 to get 45. What was the number that Alice started with?

8. Solve the following equations:
   (a) \(2(x + 1) = 6(x - 3)\)  
   (b) \(3(x + 4) = 11x\)  
   (c) \(5(x + 4) = 2(10x + 1)\)  
   (d) \(4(7 - x) = 5(x + 2)\)

9. 
   \[
   \begin{align*}
   \text{3 m} & \\
   (x + 4) \text{ m}
   \end{align*}
   \]
   (a) Write down an expression for the area of the triangle.
   (b) What is \(x\) if the area is 15 m\(^2\)?

8.3 Common Factors

As well as being able to remove brackets by expanding expressions, it is also important to be able to write expressions so that they include brackets; this is called factoring or factorisation.

Example 1

Factorise

\[4x + 6\]

Solution

First write each term as a product of factors:

\[
\begin{align*}
4x + 6 &= 2 \times 2 \times x + 2 \times 3 \\
4x + 6 &= 2(2x + 3)
\end{align*}
\]

[Note that 2 is the only factor common to both terms and is placed outside the brackets.]

Now you can check your answer by expanding it.
Example 2
Factorise

\[18n + 24\]

Solution

\[18n + 24 = 2 \times 3 \times 3 \times n + 2 \times 2 \times 2 \times 3 \]

\[= 6 (3n + 4)\]

Note that both 2 and 3 are factors of both terms, and so \(2 \times 3 = 6\) is placed outside the brackets.

Example 3
Factorise

\[4x^2 + 6x\]

Solution

\[4x^2 + 6x = 2 \times 2 \times x \times x + 2 \times 3 \times x\]

\[= 2x(2x + 3)\]

Note that both 2 and \(x\) are factors of both terms, and so \(2 \times x = 2x\) is placed outside the brackets.

Example 4
Factorise

\[5x + 20x^2\]

Solution

\[5x + 20x^2 = 5 \times x + 4 \times 5 \times x \times x\]

\[= 5x(1 + 4x)\]

Note that because 5 and \(x\) are factors of both terms, a 1 must be introduced in the bracket when the \(5x\) is placed outside the brackets.

You can check the calculation 'backwards':

\[5x(1 + 4x) = 5x \times 1 + 5x \times 4x\]

\[= 5x + 20x^2\]
Example 5

Factorise

\[3xy^2 + 12x^2y\]

Solution

\[3xy^2 + 12x^2y = 3 \times x \times y \times y + 3 \times 4 \times x \times x \times y\]

\[= 3xy(y + 4x)\]

Note that 3, x and y are factors of both terms, and so \(3 \times x \times y = 3xy\) is placed outside the brackets.

Exercises

1. Factorise:

   (a) \(2x + 4\)  
   (b) \(5x + 15\)  
   (c) \(6x + 18\)  
   (d) \(5x - 25\)  
   (e) \(3x - 21\)  
   (f) \(7x + 35\)  
   (g) \(9x - 12\)  
   (h) \(15x + 20\)  
   (i) \(42x + 15\)  

2. Factorise:

   (a) \(3x^2 + x\)  
   (b) \(5x^2 + 10\)  
   (c) \(6x - 3x^2\)  
   (d) \(6x^2 - 4x\)  
   (e) \(21x^2 + 14x\)  
   (f) \(15x - 25x^2\)  

3. Denise states that

\[4x + 6x^2 = x(4 + 6x)\]

   (a) Is her statement true?
   (b) Describe how it could be improved.

4. For each statement below, decide if it has been fully factorised and if not, complete the factorisation:

   (a) \(x^2 + x = x(x + 1)\)
   (b) \(3x^2 + 9x = 3(x^2 + 3x)\)
   (c) \(5x - 30x^2 = x(5 - 30x)\)
   (d) \(8x^2 - 32x = 4(2x^2 - 8x)\)
   (e) \(6x^2 - 18x = 3x(2x - 6)\)
   (f) \(15x - 6x^2 = 3(5x - 2x^2)\)
5. Explain why the following factorisation is incorrect:

\[ 15x + 24x^2 = 3x(5 + 24x) \]

6. Factorise:
   (a) \( xy + xz \)
   (b) \( xyz + 3yz \)
   (c) \( 4pq - 8qr \)
   (d) \( 5xyz + 20uxy \)
   (e) \( 5xy - 4py \)
   (f) \( 7xy + 12xz \)

7. Factorise:
   (a) \( x^2y + xy^2 \)
   (b) \( 3x^2y^2 + 6xy^3 \)
   (c) \( 5x^2y - 35xy \)
   (d) \( 22xy + 4xy^2 \)
   (e) \( x^2yz + xy^2z \)
   (f) \( x^2y - x^3z \)
   (g) \( x^6y^2 + xy^3 \)
   (h) \( x^4y^3 + x^2y^6 \)

8. (a) Expand \( x(x + y + z) \).
   (b) Factorise \( 5x^2 + 2xy + 4xz \).

9. Factorise:
   (a) \( 3x + 9y + 18z \)
   (b) \( 4x^2 + 2x + 8xy \)
   (c) \( 6x - 3xy + 12xz \)
   (d) \( 5xz + 20x - 35xy \)
   (e) \( 7x^2 + 14xy - 21x^2y^2 \)
   (f) \( 4x + 6xz + 15xy \)

10. Factorise:
    (a) \( 4x^2y + 12x^3y^2 + x^2 \)
    (b) \( 6x^7y^2 - 4x^5y - x^4y^2 \)
    (c) \( 3x^2y^2 - 4xy^3 + x^4y \)
    (d) \( 5x^7y - x^2y^3 + 4x^3z \)
8.4 Expansion of Two Brackets

When two brackets are multiplied together, for example,

\[(x + 2) (x + 3)\]

every term in the first bracket must be multiplied by every term in the second bracket.

Example 1
Use a table to determine

\[(x + 2) (x + 3)\]

**Solution**

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>x</td>
<td>x</td>
<td>3</td>
</tr>
<tr>
<td>(x + 2)</td>
<td>x²</td>
<td>3x</td>
</tr>
<tr>
<td>2</td>
<td>2x</td>
<td>6</td>
</tr>
</tbody>
</table>

The multiplication table is formed using the two brackets. The contents of the table give the expansion.

\[(x + 2) (x + 3) = x² + 3x + 2x + 6\]  
\[= x² + 5x + 6\]  
\[= x² + 5x + 6\]

Example 2
Use a table to determine

\[(x - 6) (x + 2)\]

**Solution**

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>(x - 6)</td>
<td>x</td>
<td>2</td>
</tr>
<tr>
<td>(x + 2)</td>
<td>x²</td>
<td>2x</td>
</tr>
<tr>
<td>-6</td>
<td>-6x</td>
<td>-12</td>
</tr>
</tbody>
</table>

So,

\[(x - 6) (x + 2) = x² + 2x - 6x - 12\]  
\[= x² - 4x - 12\]  
\[= x² - 4x - 12\]
An alternative method for expanding two brackets is shown in the next example.

Example 3
Determine
\[(x + 2)(x - 7)\]

Solution
\[(x + 2)(x - 7) = x(x - 7) + 2(x - 7)\]
\[= x^2 - 7x + 2x - 14\]
\[= x^2 - 5x - 14\]

Note how each term in the first bracket multiplies the whole of the second bracket.

Exercises
1. Copy and complete the following tables and write down each of the expansions:
   (a) \[
   \begin{array}{|c|c|c|}
   \hline
   x & x & 5 \\
   \hline
   x & 4 \\
   \hline
   \end{array}
   \]
   \[(x + 4)(x + 5)\]
   (b) \[
   \begin{array}{|c|c|c|}
   \hline
   x & x & -7 \\
   \hline
   x & 4 \\
   \hline
   \end{array}
   \]
   \[(x + 4)(x - 7)\]

   (c) \[
   \begin{array}{|c|c|c|}
   \hline
   x & x & 4 \\
   \hline
   x & -1 \\
   \hline
   \end{array}
   \]
   \[(x - 1)(x + 4)\]
   (d) \[
   \begin{array}{|c|c|c|}
   \hline
   x & x & -5 \\
   \hline
   x & -2 \\
   \hline
   \end{array}
   \]
   \[(x - 2)(x - 5)\]

2. Expand:
   (a) \[(x + 3)(x + 4)\]
   (b) \[(x - 2)(x + 5)\]
   (c) \[(x - 5)(x - 1)\]
   (d) \[(x + 7)(x - 3)\]
   (e) \[(x + 2)(x - 3)\]
   (f) \[(x + 4)(x - 1)\]
3. Expand:
   (a) \((x - 1)(x + 1)\)       (b) \((x + 2)(x - 2)\)
   (c) \((x - 5)(x + 5)\)       (d) \((x - 7)(x + 7)\)

   How are the answers to this question different from the others you have done?

4. Explain what is wrong with this statement:
   \((x + 5)^2 = x^2 + 25\)

5. Expand:
   (a) \((x + 1)^2\)           (b) \((x - 1)^2\)
   (c) \((x + 3)^2\)           (d) \((x - 5)^2\)

6. (a) Copy and complete this table:

<table>
<thead>
<tr>
<th>(\times)</th>
<th>(x)</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>2x</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

   (b) What is the expansion of
   \((2x + 1)(x + 6)\) ?

7. Expand:
   (a) \((2x + 1)(2x + 4)\)       (b) \((3x + 1)(4x + 1)\)
   (c) \((2x - 1)(3x + 2)\)       (d) \((4x - 1)(5x + 1)\)
   (e) \((2x + 1)^2\)             (f) \((4x - 3)^2\)

8. Write out the following expansions, filling in the missing terms:
   (a) \((x + 7)(x + 6) = x^2 + ? + 42\)    (b) \((x + 6)^2 = x^2 + ? + 36\)
   (c) \((x - 2)(x - 5) = x^2 + ? + 10\)    (d) \((x - 1)(2x + 1) = 2x^2 - x - ?\)
   (e) \((x + 3)(2x + 1) = ? + 7x + 3\)    (f) \((x - 7)^2 = x^2 - ? + 49\)

9. Explain what is wrong with this statement:
   \((x + 4)(x - 5) = x^2 - 20\)
10. Write out the following expansions, filling in the missing terms:
   (a) \((x + ?)(x - 1) = x^2 + x - 2\)
   (b) \((x + 4)(x - ?) = x^2 - 2x - 24\)
   (c) \((2x + 3)(x + ?) = 2x^2 + 9x + ?\)
   (d) \((x - ?)(x + 5) = x^2 - 2x - ?\)
   (e) \((x + ?)(x + ?) = x^2 + 4x + 4\)
   (f) \((x + ?)(x + ?) = x^2 + 6x + 8\)

11. The following example shows how to determine \((x + 1)^3\).

\[
\begin{array}{ccc}
\times & x & 1 \\
x & x^2 & x \\
1 & x & 1 \\
\end{array}
\]

\[
(x + 1)^3 = x^2 + x + 1
= x^2 + 2x + 1
\]

\[
\begin{array}{ccc}
\times & x^2 & 2x & 1 \\
x & x^3 & 2x^2 & x \\
1 & x^2 & 2x & 1 \\
\end{array}
\]

\[
(x + 1)^3 = (x + 1)(x^2 + 2x + 1)
= x^3 + 2x^2 + x + x^2 + 2x + 1
= x^3 + 3x^2 + 3x + 1
\]

Use the same method to determine:

(a) \((x + 1)^4\),

(b) \((x + 1)^5\).

Compare your answers with Pascal's Triangle and describe any connections that you see.
9 Arithmetic: Fractions and Percentages

9.1 Revision of Operations with Fractions

In this section we revise the basic use of fractions.

Addition
\[
\frac{a}{b} + \frac{c}{b} = \frac{a + c}{b}
\]

Note that, for addition of fractions, in this way both fractions must have the same denominator.

Multiplication
\[
\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}
\]

Division
\[
\frac{a}{b} \div \frac{c}{d} = \frac{a \times d}{b \times c}
\]

Example 1
Calculate:

(a) \( \frac{3}{5} + \frac{4}{5} \)

(b) \( \frac{3}{7} + \frac{1}{3} \)

Solution

(a) \( \frac{3}{5} + \frac{4}{5} = \frac{3 + 4}{5} = \frac{7}{5} = 1\frac{2}{5} \)

(b) \( \frac{3}{7} + \frac{1}{3} = \frac{9}{21} + \frac{7}{21} = \frac{16}{21} \) (common denominator = 21)
Example 2
Calculate:
(a) $\frac{3}{4}$ of 48
(b) $\frac{3}{5}$ of 32

Solution
(a) $\frac{3}{4}$ of 48 = $\frac{3}{4} \times 48$

= $\frac{3 \times 48}{4}$

= 36

(b) $\frac{3}{5}$ of 32 = $\frac{3}{5} \times 32$

= $\frac{3 \times 32}{5}$

= $\frac{96}{5}$

= $19 \frac{1}{5}$

Example 3
Calculate:
(a) $\frac{3}{4} \times \frac{3}{7}$
(b) $1 \frac{1}{2} \times \frac{2}{5}$

Solution
(a) $\frac{3}{4} \times \frac{3}{7} = \frac{3 \times 3}{4 \times 7}$

= $\frac{9}{28}$

(b) $1 \frac{1}{2} \times \frac{2}{5} = \frac{3}{2} \times \frac{2}{5}$

or $1 \frac{1}{2} \times \frac{2}{5} = \frac{3}{2} \times \frac{1}{\frac{5}{2}}$

= $\frac{6}{10}$

= $\frac{3}{5}$
Example 4

Calculate:

(a) \( \frac{3}{7} \div \frac{3}{4} \)

(b) \( 1\frac{3}{4} \div \frac{4}{5} \)

Solution

(a) \( \frac{3}{7} \div \frac{3}{4} = \frac{3}{7} \times \frac{4}{3} \) or \( \frac{3}{7} \div \frac{3}{4} = \frac{3 \times 4}{7} \)

\[ \frac{12}{21} \]

\[ \frac{4}{7} \]

(b) \( 1\frac{3}{4} \div \frac{4}{5} = \frac{7}{4} \div \frac{4}{5} \)

\[ \frac{7}{4} \times \frac{5}{4} \]

\[ \frac{35}{16} \]

\[ 2\frac{3}{16} \]

Exercises

1. Calculate:

(a) \( \frac{1}{7} + \frac{4}{7} \)

(b) \( \frac{3}{8} + \frac{7}{8} \)

(c) \( \frac{1}{9} + \frac{7}{9} \)

(d) \( \frac{3}{10} + \frac{1}{10} \)

(e) \( \frac{7}{13} + \frac{9}{13} \)

(f) \( \frac{6}{7} + \frac{5}{7} \)

(g) \( \frac{5}{7} - \frac{3}{7} \)

(h) \( \frac{7}{9} - \frac{4}{9} \)

(i) \( \frac{11}{13} - \frac{6}{13} \)

2. Calculate:

(a) \( \frac{1}{2} + \frac{1}{3} \)

(b) \( \frac{1}{5} + \frac{1}{7} \)

(c) \( \frac{1}{4} + \frac{1}{5} \)

(d) \( \frac{2}{3} + \frac{1}{2} \)

(e) \( \frac{7}{8} + \frac{3}{10} \)

(f) \( \frac{3}{4} + \frac{4}{5} \)

(g) \( \frac{3}{7} + \frac{2}{3} \)

(h) \( \frac{4}{9} + \frac{2}{3} \)

(i) \( \frac{1}{4} + \frac{5}{8} \)
3. Calculate:
   (a) \( \frac{1}{2} + \frac{2}{2} \)  
   (b) \( \frac{3}{4} + \frac{4}{4} \)  
   (c) \( \frac{2}{5} + \frac{3}{5} \)  
   (d) \( \frac{1}{3} + \frac{1}{2} \)  
   (e) \( \frac{4}{5} + \frac{2}{5} \)  
   (f) \( \frac{5}{7} + \frac{4}{7} \)  
   (g) \( \frac{3}{4} + \frac{2}{8} \)  
   (h) \( \frac{2}{7} + \frac{3}{3} \)  
   (i) \( \frac{5}{9} + \frac{2}{3} \)  

4. Calculate:
   (a) \( \frac{2}{2} - \frac{1}{2} \)  
   (b) \( \frac{3}{4} - \frac{1}{4} \)  
   (c) \( \frac{3}{8} - \frac{2}{4} \)  
   (d) \( \frac{5}{7} - \frac{3}{7} \)  
   (e) \( \frac{5}{8} - \frac{7}{8} \)  
   (f) \( \frac{1}{3} - \frac{1}{2} \)  
   (g) \( \frac{2}{3} - \frac{1}{9} \)  
   (h) \( \frac{3}{7} - \frac{2}{2} \)  
   (i) \( \frac{4}{4} - \frac{2}{3} \)  

5. Calculate:
   (a) \( \frac{1}{4} \) of £20  
   (b) \( \frac{1}{5} \) of 30 kg  
   (c) \( \frac{3}{4} \) of £32  
   (d) \( \frac{4}{5} \) of 90 kg  
   (e) \( \frac{5}{7} \) of 49 kg  
   (f) \( \frac{3}{8} \) of 20 m  
   (g) \( \frac{3}{5} \) of £36  
   (h) \( \frac{7}{10} \) of 42 m  

6. Calculate:
   (a) \( \frac{1}{2} \times \frac{1}{4} \)  
   (b) \( \frac{3}{8} \times \frac{1}{5} \)  
   (c) \( \frac{2}{3} \times \frac{3}{5} \)  
   (d) \( \frac{6}{7} \times \frac{2}{3} \)  
   (e) \( \frac{4}{5} \times \frac{3}{4} \)  
   (f) \( \frac{4}{7} \times \frac{3}{5} \)  
   (g) \( \frac{1}{2} \times \frac{3}{4} \)  
   (h) \( \frac{4}{9} \times \frac{3}{7} \)  
   (i) \( \frac{1}{8} \times \frac{4}{5} \)  

7. Calculate:
   (a) \( \frac{1}{2} \div \frac{1}{3} \)  
   (b) \( \frac{3}{4} \div \frac{8}{9} \)  
   (c) \( \frac{3}{5} \div \frac{4}{5} \)  
   (d) \( \frac{7}{10} \div \frac{1}{2} \)  
   (e) \( \frac{3}{4} \div \frac{3}{5} \)  
   (f) \( \frac{5}{9} \div \frac{7}{8} \)  
   (g) \( \frac{6}{7} \div \frac{2}{3} \)  
   (h) \( \frac{4}{7} \div \frac{3}{4} \)  
   (i) \( \frac{5}{6} \div \frac{2}{3} \)
8. Calculate:
   (a) \( \frac{1}{2} \times \frac{3}{4} \)  
   (b) \( \frac{3}{2} \times \frac{1}{7} \)  
   (c) \( \frac{1}{4} \times \frac{2}{3} \)  
   (d) \( \frac{1}{2} \times \frac{1}{4} \)  
   (e) \( \frac{2}{2} \times \frac{3}{4} \)  
   (f) \( \frac{2}{3} \times \frac{4}{5} \)

9. Calculate:
   (a) \( \frac{1}{2} \div \frac{3}{4} \)  
   (b) \( \frac{3}{2} \div \frac{1}{2} \)  
   (c) \( \frac{2}{4} \div \frac{2}{3} \)  
   (d) \( \frac{3}{2} \div \frac{1}{4} \)  
   (e) \( \frac{4}{2} \div \frac{4}{5} \)  
   (f) \( \frac{3}{4} \div \frac{2}{3} \)

10. Calculate:
    (a) \( \frac{1}{2} \times \frac{3}{4} \)  
    (b) \( \frac{3}{2} \times \frac{4}{7} \)  
    (c) \( \left( \frac{1}{3} \right)^2 \)

11. Calculate:
    (a) \( \frac{3}{4} \div \frac{1}{2} \)  
    (b) \( \frac{3}{2} \div \frac{1}{4} \)  
    (c) \( \frac{3}{3} \div \frac{3}{7} \)

12. Calculate:
    (a) \( \frac{4}{7} + \frac{3}{4} \)  
    (b) \( \frac{2}{2} \times \frac{3}{7} \)  
    (c) \( \frac{5}{4} - \frac{3}{6} \)  
    (d) \( \frac{6}{2} \div \frac{3}{7} \)  
    (e) \( \frac{1}{2} \times \frac{2}{3} \)  
    (f) \( \frac{2}{3} - \frac{5}{8} \)

9.2 Fractions in Context

In this section we consider the use of fractions in various contexts, and how to use the fraction key on a calculator.

Example 1

There are 600 pupils in a school. How many school lunches must be prepared if:

(a) \( \frac{3}{4} \) of the pupils have school lunches,

(b) \( \frac{2}{3} \) of the pupils have school lunches?
Solution

(a) \( \frac{3}{4} \) of 600 = \( \frac{3}{4} \times 600 \) or \( \frac{3}{4} \) of 600 = \( \frac{3}{4} \times \frac{150}{1} \times 600 \)

\[ \frac{1800}{4} = 450 \text{ lunches} \]

(b) \( \frac{2}{3} \) of 600 = \( \frac{2}{3} \times 600 \) or \( \frac{2}{3} \) of 600 = \( \frac{2}{3} \times \frac{200}{1} \times 600 \)

\[ \frac{1200}{3} = 400 \text{ lunches} \]

Example 2

The diagram opposite shows a rectangle.

(a) Calculate its \textit{perimeter}.
(b) Calculate its \textit{area}.

Solution

Perimeter = \( \frac{3}{4} + \frac{1}{3} + \frac{2}{4} + \frac{1}{3} \)

\[ 2 \frac{3}{12} + \frac{4}{12} + \frac{3}{12} + \frac{4}{12} = \frac{14}{12} = 7 \frac{1}{6} \text{ m} \]

Area = \( \frac{2}{4} \times \frac{1}{3} \)

\[ \frac{9}{4} \times \frac{4}{3} = \frac{36}{12} = 3 \text{ m}^2 \]
Example 3

A loaf of bread requires \( \frac{3}{4} \) kg of flour. How many loaves can be made from \( 6\frac{1}{2} \) kg of flour?

Solution

\[
6\frac{1}{2} \div \frac{3}{4} = \frac{13}{2} \div \frac{3}{4} = \frac{13}{2} \times \frac{4}{3} = \frac{52}{6} = 8\frac{2}{3}
\]

8 loaves can be made.

Many calculators have a key marked \( \frac{a}{bc} \), which can be used to enter fractions.

Pressing \( 2 \frac{a}{bc} 3 \) produces the display \( \frac{2}{3} \) which represents the fraction \( \frac{2}{3} \).

Pressing \( 4 \frac{a}{bc} 7 \frac{a}{bc} 9 \) produces the display \( \frac{4}{7} \frac{9}{9} \), which represents \( \frac{4}{7} \).

Note that you must write the fractions in their correct form, and not just copy the display.

(Some calculator displays may be different from this example – check the instruction booklet for your calculator.)

Exercises

1. Use your calculator to find answers for the following, making sure that they are written in the correct way:

   (a) \( \frac{1}{4} + \frac{3}{7} \)  
   (b) \( \frac{5}{7} - \frac{1}{3} \)  
   (c) \( \frac{3}{4} \div \frac{1}{9} \)  
   (d) \( \frac{1}{2} \div \frac{1}{6} \)  
   (e) \( \frac{3}{4} \times \frac{7}{8} \)  
   (f) \( \frac{4}{5} \times \frac{3}{8} \)
2. (a) Enter the fraction $\frac{6}{8}$ and then press the $\text{=}$ key on your calculator. Describe what happens.

(b) Enter the fraction $\frac{8}{6}$ and then press the $\text{=}$ key on your calculator. Describe what happens.

(c) What happens to each of the fractions listed below if you enter it into your calculator and then press the $\text{=}$ key:

\[
\frac{3}{7}, \frac{9}{2}, \frac{4}{6}, \frac{6}{4}, \frac{10}{3}, \frac{3}{10}
\]

3. Calculate the area and perimeter for each of the rectangles below:

(a) \[ \begin{array}{c}
\frac{2}{3} \text{ m} \\
\frac{1}{2} \text{ m}
\end{array} \]

(b) \[ \begin{array}{c}
2\frac{3}{5} \text{ m} \\
\frac{1}{4} \text{ m}
\end{array} \]

4. A school has 800 pupils. The Headteacher decides to send a questionnaire to $\frac{2}{5}$ of the pupils. How many pupils receive a questionnaire?

5. A firm that makes floppy discs knows that $\frac{1}{20}$ of the discs they produce have faults. How many faulty floppy discs would you have if you bought:

(a) 100 discs, (b) 80 discs, (c) 2000 discs?

6. A cake recipe requires $\frac{3}{8}$ kg of flour. How many cakes could be made with:

(a) 3 kg flour, (b) 6 kg flour, (c) $\frac{2}{3}$ kg flour,

(d) 1 kg flour, (e) $\frac{1}{2}$ kg flour, (f) $\frac{1}{3}$ kg flour.

7. The rectangle opposite has an area of $2\frac{3}{5}$ cm$^2$.

What is the length, $x$, of the rectangle?
8. Sheets of paper are \( \frac{1}{80} \) cm thick. Calculate the height of a pile of paper that contains:

(a) 40 sheets,  
(b) 120 sheets,  
(c) 70 sheets,  
(d) 140 sheets.

How many sheets would there be in a pile of paper \( 4 \frac{1}{2} \) cm high?

9. A bottle contains \( 1 \frac{2}{5} \) litres of orange squash. To make one drink, \( \frac{1}{200} \) of a litre of squash is needed.

How many drinks can be made from the bottle of squash?

10. Calculate the volume of the following cuboid:

![Cuboid Diagram]

\[
\begin{align*}
\text{Volume} &= \text{length} \times \text{width} \times \text{height} \\
&= \left(4 \frac{1}{5}\right) \times \left(3 \frac{3}{8}\right) \times \left(2 \frac{3}{4}\right) \\
&= \left(\frac{21}{5}\right) \times \left(\frac{27}{8}\right) \times \left(\frac{11}{4}\right) \\
&= \frac{21 \times 27 \times 11}{5 \times 8 \times 4} \\
&= \frac{7347}{160} \\
&= 46.04375 \text{ cm}^3
\end{align*}
\]

9.3 Conversion of Fractions and Percentages

To convert a fraction to a percentage, multiply by 100.

To convert a percentage to a fraction, divide by 100 or multiply by \( \frac{1}{100} \).

Example 1

Convert the following fractions to percentages:

(a) \( \frac{17}{100} \)  
(b) \( \frac{9}{10} \)  
(c) \( \frac{3}{5} \)  
(d) \( \frac{3}{4} \)  
(e) \( \frac{1}{3} \)  
(f) \( \frac{1}{8} \)
Solution

(a) \[
\frac{17}{100} \times 100 = \frac{1700}{100} = 17\%
\]

(b) \[
\frac{9}{10} \times 100 = \frac{900}{10} = 90\%
\]

(c) \[
\frac{3}{5} \times 100 = \frac{300}{5} = 60\%
\]

(d) \[
\frac{3}{4} \times 100 = \frac{300}{4} = 75\%
\]

(e) \[
\frac{1}{3} \times 100 = \frac{100}{3} = 33\frac{1}{3}\%
\]

(f) \[
\frac{1}{8} \times 100 = \frac{100}{8} = 12\frac{4}{8} = 12\frac{1}{2}\%
\]

Example 2

Convert these percentages to fractions:

(a) 30%  
(b) 80%  
(c) 45%  
(d) 6%  
(e) 16\frac{1}{2}\%  
(f) 62\frac{1}{2}\%

Solution

(a) 30% = \[
\frac{30}{100} = \frac{3}{10}
\]
(b) \[ 80\% = \frac{80}{100} = \frac{8}{10} \]

(c) \[ 45\% = \frac{45}{100} = \frac{9}{20} \]

(d) \[ 6\% = \frac{6}{100} = \frac{3}{50} \]

(e) \[ 16\frac{1}{2}\% = 16\frac{1}{2} \times \frac{1}{100} = \frac{33}{2} \times \frac{1}{100} = \frac{33}{200} \]

(f) \[ 62\frac{1}{2}\% = 62\frac{1}{2} \times \frac{1}{100} = \frac{125}{2} \times \frac{1}{100} = \frac{125}{200} = \frac{5}{8} \]

Example 3

A football team is based on a squad of 20 players. In one season 8 players are shown a red or yellow card.

(a) What percentage of the squad is shown a red or yellow card?

(b) What percentage of the squad is not shown a red or yellow card?

Solution

(a) \[ \frac{8}{20} \times 100 = \frac{800}{20} = 40\% \]

(b) \[ 100 - 40 = 60\% \]
Exercises

1. Convert the following percentages to fractions:
   (a) 50%  
   (b) 75%  
   (c) 40%  
   (d) 25%  
   (e) 20%  
   (f) 10%  
   (g) 8%   
   (h) 58%  
   (i) 36%  
   (j) 64%  
   (k) 76%  
   (l) 12%

2. Convert the following fractions to percentages:
   (a) $\frac{7}{10}$  
   (b) $\frac{1}{2}$   
   (c) $\frac{1}{4}$  
   (d) $\frac{3}{4}$  
   (e) $\frac{7}{20}$  
   (f) $\frac{6}{25}$  
   (g) $\frac{19}{20}$  
   (h) $\frac{17}{25}$  
   (i) $\frac{3}{5}$   
   (j) $\frac{1}{5}$   
   (k) $\frac{11}{20}$  
   (l) $\frac{7}{50}$

3. Convert the following percentages to fractions:
   (a) $12\frac{1}{2}$%  
   (b) $66\frac{2}{3}$%  
   (c) $33\frac{1}{3}$%  
   (d) $14\frac{1}{2}$%  
   (e) $18\frac{1}{2}$%  
   (f) $4\frac{1}{4}$%

4. Convert these fractions to percentages:
   (a) $\frac{1}{8}$  
   (b) $\frac{1}{6}$  
   (c) $\frac{3}{8}$  
   (d) $\frac{47}{200}$  
   (e) $\frac{61}{200}$  
   (f) $\frac{2}{3}$

5. In a class of 25 pupils there are 8 individuals who play in the school hockey team. What percentage of the class play in the hockey team?

6. Halim scores 32 out of 80 in a test. Express his score as a percentage.
7. An athlete has completed 250 m of a 400 m race. What percentage of the distance has the athlete run?

8. A double decker bus has 72 seats; there are 18 empty seats on the bus.
   (a) What percentage of the seats are empty?
   (b) What percentage of the seats are being used?

9. Andy buys a bag of 12 apples at a supermarket; there are 4 bruised apples in the bag.
   (a) What percentage of the apples are bruised?
   (b) What percentage of the apples are not bruised?

10. Jason took 4 tests at school and his results are given below:
    
    | Subject   | Score | Total |
    |-----------|-------|-------|
    | Science   | 60    | 80    |
    | Maths     | 75    | 100   |
    | English   | 38    | 50    |
    | French    | 28    | 40    |
    
    (a) Express his score for each test as a percentage.
    (b) Write down his average percentage score for the 4 tests.

### 9.4 Finding Percentages

In this section we revise finding percentages of quantities.

#### Example 1

Calculate 20% of £120.

**Solution**

\[
20\% \text{ of } £120 = \frac{20}{100} \times 120 \\
= \frac{2}{10} \times 120 \\
= £24
\]
Example 2
Calculate 75% of 48 kg.

Solution

\[
75\% \text{ of } 48 \text{ kg} = \frac{75}{100} \times 48
\]
\[
= \frac{3}{4} \times 48
\]
\[
= 36 \text{ kg}
\]

Value Added Tax (VAT) is added to the price of many products; in the UK it is currently 17\(\frac{1}{2}\)%.

An interesting way to calculate 17\(\frac{1}{2}\)% is to use the fact that

\[17\frac{1}{2} = 10 + 5 + 2\frac{1}{2};\] this is illustrated in the next example.

Example 3
A bike costs £180 before VAT is added. How much VAT must be added to the cost of the bike, if VAT is charged at 17\(\frac{1}{2}\)%?

Solution

\[
10\% \text{ of } £180 = £18
\]
\[
5\% \text{ of } £180 = £9
\]
\[
2\frac{1}{2}\% \text{ of } £180 = £4.50
\]
\[
17\frac{1}{2}\% \text{ of } £180 = £18 + £9 + £4.50
\]
\[
= £31.50
\]
Exercises

1. Calculate:
   (a) 50% of £22   (b) 10% of 70 m   (c) 25% of £60
   (d) 30% of 80 m  (e) 60% of £40   (f) 90% of 50 kg
   (g) 75% of £30   (h) 25% of 6 kg    (i) 30% of 32 kg
   (j) 16% of £40   (k) 70% of 8 m    (l) 35% of £20

2. Use the method of Example 3 to calculate the VAT that must be added to the following prices at a rate of $17\frac{1}{2}$%:
   (a) £200   (b) £300   (c) £40
   (d) £30    (e) £28    (f) £38

3. (a) Calculate $17\frac{1}{2}$% of £25
   (b) Describe the most sensible way to give your answer.

4. Calculate $17\frac{1}{2}$% of the following amounts, giving your answers to a sensible degree of accuracy:
   (a) £15   (b) £75   (c) £7

5. Use a method similar to Example 3 to calculate 15% of £120.

6. A computer costs £900, but $17\frac{1}{2}$% VAT must be added to this price.
   (a) Calculate $17\frac{1}{2}$% of £900.
   (b) Calculate the total cost of the computer.

7. A company employs 240 staff. The number of staff is to be increased by 20%. How many new staff will the company employ?

8. A bike costs £186. The price is to be reduced by $33\frac{1}{3}$% in a sale.
   (a) Calculate how much you would save by buying the bike in the sale.
   (b) How much would you pay for the bike in the sale?

9. In a school there are 280 pupils in Year 7. 85% of these pupils go on a trip to Alton Towers. How many pupils go on the trip?

10. Alec scores 75% on a test with a maximum of 56 marks. How many marks does Alec score in the test?
9.5 Increasing and Decreasing Quantities by a Percentage

When increasing or decreasing by a percentage there are two possible approaches: one is to find the actual increase or decrease and to add it to, or subtract it from, the original amount. The second approach is to use a simple multiplication. For example, to increase by 20%, multiply by 1.2. We can illustrate this by considering a price, say £p, that increases by 20%.

The increase is $\pounds p \times \frac{20}{100} = \pounds 0.2p$
so the new price is

$\pounds p + \pounds 0.2p = \pounds (1 + 0.2)p$

$= \pounds 1.2p$

and we can see that a 20% increase is equivalent to multiplying by 1.2 to get the new price.

Note that

$100\% + 20\% = 120\% \Rightarrow \frac{120}{100} = 1.2$

Similarly, a decrease of 20% is equivalent to

$100\% - 20\% = 80\% \Rightarrow \frac{80}{100} = 0.8$

i.e. a multiplication by 0.8.

Example 1

The price of a jar of coffee is £2.00. Calculate the new price after an increase of 10%.

Solution

10% of £2.00 = $\frac{10}{100} \times 2$ or $100\% + 10\% = 110\%$, so multiply by 1.1

New price = $2 + 0.2$ New price = $1.1 \times £2$

= £2.20

= £2.20
Example 2
In a sale, the price of a TV is reduced by 40%. What is the sale price if the original price was £170.

Solution
\[
40\% \text{ of } £170 = \frac{40}{100} \times 170 \quad \text{or} \quad 100\% - 40\% = 60\%,
\]
\[
= £68 \quad \text{so multiply by } 0.6
\]
Sale price \( = 170 - 68 \quad \text{Sale price} \quad = 0.6 \times 170 \)
\[
= £102
\]

Example 3
Jared earns £24 each week by working in a shop. His wages are to be increased by 5%. How much will he then earn each week?

Solution
\[
5\% \text{ of } £24 = \frac{5}{100} \times 24 \quad \text{or} \quad 100\% + 5\% = 105\%,
\]
\[
= £1.20 \quad \text{so multiply by } 1.05
\]
New wages \( = 24 + 1.20 \quad \text{New wages} \quad = 1.05 \times 24 \)
\[
= £25.20
\]

Exercises
1. Add 10% to:
   (a) £40  (b) £136  (c) £262

2. Reduce the following prices by 20%:
   (a) £50  (b) £92  (c) £340

3. (a) Increase 40 m by 30%  (b) Increase £60 by 5%
   (c) Increase £66 by 20%  (d) Increase 80 kg by 40%
   (e) Increase £1000 by 30%  (f) Decrease £60 by 25%
   (g) Reduce 70 kg by 5%  (h) Reduce £90 by 15%
   (i) Increase 40 m by 7%  (j) Increase £18 by 4%
4. A computer costs £600. In a sale there is a 20% discount on the price of the item. Calculate the sale price of the computer.

5. A shopkeeper increases all the prices in his shop by 4%. What is the new price of each of the items below? Give your answers to the nearest penny.

<table>
<thead>
<tr>
<th>Item</th>
<th>Old Price</th>
<th>New Price</th>
</tr>
</thead>
<tbody>
<tr>
<td>Box of chocolates</td>
<td>£3</td>
<td>£3.42</td>
</tr>
<tr>
<td>Bag of flour</td>
<td>75p</td>
<td>78p</td>
</tr>
<tr>
<td>Packet of sweets</td>
<td>50p</td>
<td>52p</td>
</tr>
<tr>
<td>Tin of beans</td>
<td>20p</td>
<td>20.8p</td>
</tr>
<tr>
<td>Can of drink</td>
<td>45p</td>
<td>47.1p</td>
</tr>
</tbody>
</table>

6. A CD player costs £90. In a sale the price is reduced by 25%. Calculate the sale price.

7. A certain type of calculator costs £8. A teacher buys 30 of these calculators for her school and is given a 20% discount. How much does she pay in total?

8. Add $\frac{11}{2}$% VAT to the following prices, giving your answers to the nearest penny:

   (a) £400
   (b) £22
   (c) £65

9. The population of a town is 120 000. What is the total population of the town after a 5% increase?

10. Hannah invests £800 in a building society. Every year 5% interest is added to her money.

    (a) Explain why, after 2 years she has £882 in her account.
    (b) How much money does she have after 5 years? (Give your answer to the nearest penny.)

11. Andrew has £100 to invest in a building society. At the end of each year, 10% interest is added to his investment.

    (a) What is the multiplier that can be used each year to calculate the new amount in the account?
    (b) Show that the multiplier for 2 years is 1.21.
    (c) What is the multiplier for $n$ years?
    (d) How many years does it take to $\textit{double}$ the £100 investment?
9.6 Finding the Percentage Increase and Decrease

When a quantity increases, we can find the percentage increase using this formula:

\[
\text{Percentage increase} = \frac{\text{increase}}{\text{original amount}} \times 100
\]

Similarly,

\[
\text{Percentage decrease} = \frac{\text{decrease}}{\text{original amount}} \times 100
\]

Example 1
The price of a drink increases from 40p to 45p. What is the percentage increase?

Solution
Increase \(= 45p - 40p = 5p\)

Percentage increase \(= \frac{5}{40} \times 100 = \frac{25}{2} = 12.5\%\)

Example 2
The number of pupils in a school increases from 820 to 861. Calculate the percentage increase.

Solution
Increase \(= 861 - 820 = 41\) pupils

Percentage increase \(= \frac{41}{820} \times 100 = 5\%\)
Example 3
Although the lion is thought of as an African animal, there is a small population in India and elsewhere in Asia. The number of lions in India decreased from 6000 to 3900 over a 10-year period. Calculate the percentage decrease in this period.

Solution
Decrease = 6000 – 3900
= 2100 lions

Percentage decrease = \frac{2100}{6000} \times 100
= 35%

Example 4
The price of cheese, per kg, is increased from £3.26 to £3.84. What is the percentage increase?

Solution
Increase = £3.84 – £3.26
= £0.58

Percentage increase = \frac{0.58}{3.26} \times 100
= 17.8% to 1 decimal place

Note: You might find it easier to work through the example in pence, but note that all quantities must be expressed in pence.

Increase = (384 – 326)p
= 58p

Percentage increase = \frac{58}{326} \times 100
= 17.8% to 1 decimal place

Example 5
In a sale, the price of a bike is reduced from £180 to £138. Calculate the percentage reduction in price, correct to 1 decimal place.
Solution

Reduction = 180 − 138
= £42

Percentage reduction = \( \frac{42}{180} \times 100 \)
= 23.3\% to 1 decimal place.

Exercises

1. The price of a school lunch increases from £1.40 to £1.54. Calculate the percentage increase in the price.

2. A television priced at £500 is reduced in price to £400 in a sale. Calculate the percentage reduction in the price of the television.

3. The price of a car increases from £8000 to £8240. What is the percentage increase in the price of the car?

4. A shopkeeper buys notepads for 60p each and sells them for 80p each. What percentage of the selling price is profit?

5. The value of an antique clock increases from £300 to £345. Calculate the percentage increase in the value of the clock.

6. The number of books in a school library is increased from 2220 to 2354. What is the percentage increase in the number of books?

7. The height of a tomato plant increases from 80 cm to 95 cm. Calculate the percentage increase in the height, correct to 1 decimal place.

8. The price of a bus fare is reduced from 55p to 40p. Calculate the percentage reduction in the price of the bus fare, correct to 1 decimal place.

9. The mass of a person on a diet decreases from 75 kg to 74 kg. Calculate the percentage reduction in their mass, correct to 3 significant figures.
10. Jasmine invests £250 in a building society. After the first year her account contains £262.50. After the second year it contains £280.88. Calculate the percentage increase of the amount in her account:

(a) during the first year,
(b) during the second year,
(c) over the two years.

Give your answers correct to 2 decimal places.

9.7 Reverse Percentage Calculations

The process of adding a percentage to a quantity can be reversed.

For example, if the cost of a portable TV is £141 including $17\frac{1}{2}\%$ VAT, the cost before adding the VAT can be found. The multiplier in this example is 1.175, as the price is made up of $100\% + 17.5\% = 117.5\%$, which is equivalent to multiplying by

$$\frac{117.5}{100} = 1.175$$

Original price $\times 1.175 \rightarrow £141$

$£120 \div 1.175 \rightarrow £141$

Example 1

Jane's salary was increased by 10\% to £9350. What was her original salary?

Solution

$100\% + 10\% = 110\%$,

which $= \frac{110}{100} = 1.1$

Therefore Jane's original salary would have been multiplied by 1.1 to give £9350. So to calculate her original salary, divide by 1.1.

Original salary $\times 1.1 \rightarrow £9350$

£8500 $\div 1.1 \rightarrow £9350$
Example 2
In a sale, the price of a video recorder is reduced by 22% to £218.40. How much money would you save by buying the video recorder in the sale?

Solution

\[
100\% - 22\% = 78\% \\
= \frac{78}{100} \\
= 0.78
\]

The original price would have been multiplied by 0.78 to get the sale price. So divide by 0.78 to find the original price.

Original price \times 0.78 \rightarrow £218.40

£280 \div 0.78 \rightarrow £218.40

Saving = Original price – Sale price

= £280 – £218.40

= £61.60

Example 3
The cost of an order, including VAT at \(17\frac{1}{2}\%\), is £274.95. Calculate the cost of the order \textit{without} VAT.

Solution

Original cost \times 1.175 \rightarrow £274.95

£234 \div 1.175 \rightarrow £274.95

Cost of the order without VAT is £234.00.
Exercises

1. In a sale the prices of all the clothes in a shop are reduced by 20%. Calculate the original prices of the items below:

<table>
<thead>
<tr>
<th>Item</th>
<th>Sale Price</th>
</tr>
</thead>
<tbody>
<tr>
<td>Jeans</td>
<td>£36</td>
</tr>
<tr>
<td>Coat</td>
<td>£56</td>
</tr>
<tr>
<td>Shirt</td>
<td>£14</td>
</tr>
</tbody>
</table>

2. The price of a car is increased by 4% to £12 480. What was the original price?

3. The amount that Jason earns for his paper round is increased by 2% to £21.93 per week. How much extra money does Jason now get each week?

4. A special value packet of breakfast cereal contains 25% more than the standard packet. The special value packet contains 562.5 grams of cereal. How much does the standard packet contain?

5. The bill for repairing a computer is £29.38 which includes VAT at $17\frac{1}{2}$. What was the bill before the VAT was added?

6. The height of a plant increases by 18%, to 26 cm. Calculate the original height of the plant, correct to the nearest cm.

7. A 3.5% pay rise increases Mr Smith's annual salary to £21 735. What was his original salary?

8. The price of a bike in a sale is £145. If the original price has been reduced by 12$\frac{1}{2}$%, what was the original price? (Give your answer to the nearest pence.)

9. Alice carries out an experiment to record how quickly plants grow. One plant increases in height from 12.0 cm to 13.8 cm in one week. A second plant increases by the same percentage to 16.1 cm. What was the original height of the second plant?

10. James buys a computer. The seller reduces the price by 30% and adds VAT at 17.5%. If James pays £1551 for the computer, what was its original price? (Give your answer to the nearest pence.)
10 Probability - Two Events

10.1 Recap: Basic Probability for One Event

In this section we revise the use of probabilities for single events, remembering that:

\[
\text{Probability of an event} = \frac{\text{number of successful outcomes}}{\text{number of possible outcomes}}
\]

Example 1

A tube of sweets contains 10 red sweets, 7 blue sweets, 8 green sweets and 5 orange sweets. If a sweet is taken at random from the tube, what is the probability that it is:

(a) red,
(b) orange,
(c) green or red,
(d) not blue?

Solution

There are 30 sweets in the tube.

(a) There are 10 red sweets in the tube, so

\[
p(\text{red}) = \frac{10}{30} = \frac{1}{3}
\]

(b) There are 5 orange sweets in the tube, so

\[
p(\text{orange}) = \frac{5}{30} = \frac{1}{6}
\]
(c) There are 8 green sweets and 10 red sweets in the tube, so

\[ p \text{ (green or red)} = \frac{8 + 10}{30} = \frac{18}{30} = \frac{3}{5} \]

(d) There are 23 sweets that are not blue in the tube, so

\[ p \text{ (not blue)} = \frac{23}{30} \]

Example 2

Nine balls, each marked with a number from 1 to 9, are placed in a bag and one ball is taken out at random. What is the probability that the number on the ball is:

(a) odd,
(b) a multiple of 3,
(c) a 5,
(d) not a 7?

Solution

There are 9 possible outcomes in each case.

(a) There are 5 possible odd numbers, so

\[ p \text{ (odd)} = \frac{5}{9} \]

(b) There are 3 numbers that are multiples of 3, so

\[ p \text{ (multiple of 3)} = \frac{3}{9} = \frac{1}{3} \]

(c) There is only 1 ball numbered 5, so

\[ p \text{ (5)} = \frac{1}{9} \]

(d) There are 8 numbers that are not 7, so

\[ p \text{ (not 7)} = \frac{8}{9} \]
Exercises

1. There are 16 girls and 8 boys in the tennis club. One of these is chosen at random to enter a competition. What is the probability that a girl is chosen?

2. A bag contains 8 blue balls, 7 green balls and 5 red balls. A ball is taken at random from the bag. What is the probability that the ball is:
   (a) red,
   (b) blue,
   (c) green,
   (d) yellow?

3. A card is taken at random from a standard 52-card pack of playing cards. What is the probability that it is:
   (a) a seven,
   (b) a heart,
   (c) a red card,
   (d) a red six?

4. If you roll a fair dice, what is the probability that the number you get is:
   (a) 5
   (b) an odd number,
   (c) a number greater than 1,
   (d) a multiple of 4?

5. Ishmail writes a computer program that produces at random one of the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9. What is the probability that the program produces:
   (a) an even number,
   (b) a multiple of 4,
   (c) a number less than 7,
   (d) a multiple of 5?

6. The police line up 10 people in an identity parade; only one of the people is the criminal. A witness does not recognise the criminal and so chooses a person at random. What is the probability that:
   (a) the criminal is chosen,
   (b) the criminal is not chosen?
7. There are 18 boys and 17 girls in a class. One of these pupils is selected at random to represent the class. What is the probability that the pupil selected is a girl?

8. In Hannah’s purse there are three £1 coins, five 10p coins and eight 2p coins. If she takes a coin at random from her purse, what is the probability that it is:
   (a) a £1 coin,
   (b) a 2p coin,
   (c) not a £1 coin,
   (d) a £1 coin or a 10p coin?

9. Some of the children in a class write down the first letter of their surname on a card; these cards are shown below:

   W M G S S J
   W S H E H S E A
   M T H S E I

   (a) One of these cards is taken at random. What is the probability that the letter on it is:
      (i) W,
      (ii) S or T,
      (iii) J or M,
      (iv) not H
      (v) a vowel?

   (b) Which letter is the most likely to be chosen?

10. Rachel buys a new CD, on which is her favourite track, 8 other tracks she likes and 2 tracks that she does not like. She sets her CD player to play at random. What is the probability that the first track it plays is:
    (a) Rachel’s favourite,
    (b) a track that she likes,
    (c) a track that she does not like?
10.2 Outcomes with Two Events

When two events take place at the same time, it is important to list all the possible outcomes in some way. There are three possible approaches: systematic listing, using a table or using a tree diagram.

Example 1

Caitlin and Dave each buy a chocolate bar from a vending machine that sells Aero, Bounty, Crunchie and Dime bars.

List the possible pairs of bars which Caitlin and Dave can choose.

Solution

<table>
<thead>
<tr>
<th>Caitlin</th>
<th>Dave</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>A</td>
</tr>
<tr>
<td>A</td>
<td>B</td>
</tr>
<tr>
<td>A</td>
<td>C</td>
</tr>
<tr>
<td>A</td>
<td>D</td>
</tr>
<tr>
<td>B</td>
<td>A</td>
</tr>
<tr>
<td>B</td>
<td>B</td>
</tr>
<tr>
<td>B</td>
<td>C</td>
</tr>
<tr>
<td>B</td>
<td>D</td>
</tr>
<tr>
<td>C</td>
<td>A</td>
</tr>
<tr>
<td>C</td>
<td>B</td>
</tr>
<tr>
<td>C</td>
<td>C</td>
</tr>
<tr>
<td>C</td>
<td>D</td>
</tr>
<tr>
<td>D</td>
<td>A</td>
</tr>
<tr>
<td>D</td>
<td>B</td>
</tr>
<tr>
<td>D</td>
<td>C</td>
</tr>
<tr>
<td>D</td>
<td>D</td>
</tr>
</tbody>
</table>

A = Aero  
B = Bounty  
C = Crunchie  
D = Dime
Example 2
A fair dice is rolled and an unbiased coin is tossed. Draw a table to show the possible outcomes.

Solution

<table>
<thead>
<tr>
<th>COIN</th>
<th>DICE</th>
</tr>
</thead>
<tbody>
<tr>
<td>H</td>
<td>H1</td>
</tr>
<tr>
<td></td>
<td>H2</td>
</tr>
<tr>
<td>T</td>
<td>T1</td>
</tr>
<tr>
<td></td>
<td>T2</td>
</tr>
<tr>
<td></td>
<td>H3</td>
</tr>
<tr>
<td></td>
<td>H4</td>
</tr>
<tr>
<td></td>
<td>H5</td>
</tr>
<tr>
<td></td>
<td>H6</td>
</tr>
</tbody>
</table>

The table shows that there are 12 possible outcomes.

Example 3
Draw a table to show all the possible total scores when two fair dice are thrown at the same time.

Solution

<table>
<thead>
<tr>
<th>DICE</th>
<th>B</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>5</td>
<td>6</td>
</tr>
<tr>
<td>6</td>
<td>7</td>
</tr>
</tbody>
</table>

The table shows that there are 36 possible outcomes, and gives the total score for each outcome.

Example 4
Use a tree diagram to show the possible outcomes when two unbiased coins are tossed.

Solution

The diagram shows that there are 4 possible outcomes.
Example 5
In a drawer there are some white socks and some black socks. Tim takes out one sock and then a second. Draw a tree diagram to show the possible outcomes.

Solution
There are four possible outcomes, of which two will produce two socks of the same colour.

Exercises
1. Copy and complete the table to show all possible outcomes when 2 fair coins are tossed.

<table>
<thead>
<tr>
<th>OUTCOMES</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>B</td>
<td>B</td>
</tr>
<tr>
<td></td>
<td>B</td>
<td>W</td>
</tr>
<tr>
<td></td>
<td>W</td>
<td>B</td>
</tr>
<tr>
<td></td>
<td>W</td>
<td>W</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>COIN A</th>
<th>H</th>
<th>T</th>
</tr>
</thead>
<tbody>
<tr>
<td>H</td>
<td></td>
<td></td>
</tr>
<tr>
<td>T</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>B B</th>
<th>B W</th>
<th>W B</th>
<th>W W</th>
</tr>
</thead>
</table>

2. Two spinners are numbered 1 to 4 as shown in the diagram:

(a) Copy and complete the table below, to show all possible outcomes when they are spun, writing the total score for each outcome.

<table>
<thead>
<tr>
<th>SPINNER B</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>S P I N A R</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
</tr>
</tbody>
</table>

(b) What is the total number of possible outcomes?
(c) How many outcomes give a score of 5?
3. Two fair dice are renumbered using $-2, -1, 0, 1, 2, 3$ instead of the usual numbers. The two dice are thrown in the normal way.
(a) Draw a table to show the total score for each of the possible outcomes.
(b) How many ways are there of scoring 0?

4. The two spinners shown in the diagram opposite, are spun at the same time:
(a) Draw a table to show all possible outcomes, and the total score for each outcome.
(b) How many different outcomes are there?
(c) How many outcomes give a score of 6?

5. In a bag there are red and blue counters. Two counters are taken out of the bag at random.
(a) Copy and complete the tree diagram below, to show all outcomes:

(b) How many outcomes include a red counter?
(c) How many outcomes include a blue counter?

6. (a) Draw a tree diagram to show all possible outcomes when two unbiased coins are tossed.
(b) Extend your tree diagram to show the possible outcomes when three unbiased coins are tossed.
(c) How many outcomes are there when three unbiased coins are tossed?
(d) How many outcomes are there when four unbiased coins are tossed?

7. In a jar there are three different types of sweets, eclairs, mints and toffees; two sweets are taken at random.
(a) Draw a tree diagram to show the possible outcomes.
(b) How many of the outcomes include a toffee?
(c) How many of the outcomes include a mint and a toffee?
8. A red dice, a blue dice and a green dice are put into a bag; all the dice are fair. One is then taken out and rolled. The colour of the dice and the score shown are recorded.
   (a) How many possible outcomes are there?
   (b) How many outcomes include a 5?

9. In a game, two fair dice are rolled and the scores are multiplied together.
   (a) Draw a table to show the possible outcomes and their scores.
   (b) How many ways are there of scoring 12?
   (c) How many ways are there of scoring 18?

10. A bag contains a mixture of red, green and white balls. Three balls are taken at random from the bag.
    (a) Write down all possible outcomes.
    (b) How many outcomes include a red ball?
    (c) How many outcomes include a red or a white ball?
    (d) How many outcomes include a red and a green ball?

10.3 Probability Using Listings

When the outcomes for two events are equally likely, the probabilities of particular outcomes can be found.

Example 1
Look at the list of chocolate bars which can be chosen by Caitlin and Dave in Example 1 of section 10.2. What is the probability that they both choose the same type of chocolate bar?

Solution
There are 16 different outcomes and all are equally likely.
In 4 of these outcomes both Caitlin and Dave choose the same type of bar.
So
\[ p(\text{same type}) = \frac{4}{16} \text{ or } \frac{1}{4} \]
Example 2
When two unbiased coins are tossed, determine the probability of obtaining:
(a) two heads,
(b) two tails,
(c) a head and a tail.

Solution
The table shows the possible outcomes:
In this situation there are 4 outcomes that are equally likely.

<table>
<thead>
<tr>
<th></th>
<th>H</th>
<th>T</th>
</tr>
</thead>
<tbody>
<tr>
<td>H</td>
<td>H</td>
<td>H</td>
</tr>
<tr>
<td>T</td>
<td>T</td>
<td>H</td>
</tr>
</tbody>
</table>

(a) Here 1 of the 4 outcomes gives 2 heads, so
\[p (2 \text{ heads}) = \frac{1}{4}\]

(b) Here 1 of the 4 outcomes gives 2 tails, so
\[p (2 \text{ tails}) = \frac{1}{4}\]

(c) Here 2 of the outcomes gives a head and a tail, so
\[p (\text{head and a tail}) = \frac{2}{4} = \frac{1}{2}\]

Example 3
Two fair dice are rolled at the same time. What is the probability that the total score is:
(a) 6,
(b) greater than 9,
(c) less than 7?

Solution
The table shows the possible outcomes.
There are 36 equally likely scores.

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
<td>9</td>
</tr>
<tr>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
<td>9</td>
<td>10</td>
</tr>
<tr>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
<td>9</td>
<td>10</td>
<td>11</td>
</tr>
<tr>
<td>6</td>
<td>7</td>
<td>8</td>
<td>9</td>
<td>10</td>
<td>11</td>
<td>12</td>
</tr>
</tbody>
</table>

(a) There are 5 outcomes that give a score of 6, so
\[p (6) = \frac{5}{36}\]
(b) There are 6 outcomes that give a score greater than 9, so
\[ p \text{ (greater than 9)} = \frac{6}{36} = \frac{1}{6} \]

(c) There are 15 outcomes that give scores of less than 7, so
\[ p \text{ (less than 7)} = \frac{15}{36} = \frac{5}{12} \]

Exercises

1. Use information from the table in Example 3 to answer this question:
When two fair dice are thrown, what is the probability that the total score is:
   (a) 9,  
   (b) an odd number,  
   (c) greater than 10,  
   (d) less than 8 ?

2. The diagram shows two spinners which are both spun.
   What is the probability that the total score on the two spinners is:
   (a) 7,  
   (b) 6,  
   (c) greater than 10,  
   (d) less than 5 ?

3. An unbiased coin is tossed and a fair dice is thrown. Use a table of outcomes to determine the probability of each of the following:
   (a) obtaining a head and a 3,  
   (b) obtaining a tail and an even number,  
   (c) obtaining a head and a prime number.

4. The two spinners shown in the diagram are both spun.
   (a) Draw up a table to show the possible outcomes.  
   (b) What is the probability that both spinners show the same colour?  
   (c) What is the probability of obtaining a yellow and a red?  
   (d) What is the probability of obtaining a red and a blue?
5. The diagram shows two spinners that are spun at the same time:

![Spinners Diagram]

Use a table to determine the probability of obtaining a total score of:
(a) 6  (b) 0  (c) 1  (d) 3

6. For the spinners in question 5, determine the probability of obtaining a total score that is:
(a) an even number,
(b) greater than 1,
(c) less than 1,
(d) less than 6.

7. Two unbiased coins are tossed at the same time. What is the probability of obtaining:
(a) at least one head,
(b) no heads?

8. Three unbiased coins are tossed at the same time. Use a tree diagram to show the outcomes and determine the probability of obtaining:
(a) 3 heads,
(b) at least 1 head,
(c) at least 2 heads.

9. Two fair dice are rolled and the scores on each dice are multiplied together to give a total score. What is the probability of getting a total score:
(a) of 12,
(b) of 20,
(c) greater than 25,
(d) less than 30,
(e) that is an even number?

10. If 4 unbiased coins are tossed at the same time, what is the probability of obtaining the same number of heads as tails?
10.4 Multiplication Law for Independent Events

Probabilities can be assigned to tree diagrams, and then multiplication can be used to determine the probabilities for combined events.

\[
\begin{array}{c|c|c}
\text{OUTCOMES} & \text{PROBABILITIES} \\
\hline
p(A) & A & p(A) \\
p(A) & A & p(A) \\
\hline
p(B) & A & p(A) \\
p(B) & A & p(B) \\
\hline
p(B) & B & p(B) \\
p(B) & B & p(B) \\
\end{array}
\]

Note: Here we have an experiment with two possible outcomes, A and B, and the experiment is repeated once. It is assumed that the probability of either A or B remains the same when the experiment is repeated; in this case, we say that A and B are independent events.

Example 1

Two fair dice are rolled. Use a tree diagram to determine the probability of obtaining:

(a) 2 sixes,  
(b) 1 six,  
(c) no sixes.

Solution

The tree diagram is shown below:

\[
\begin{array}{c|c|c}
\text{OUTCOMES} & \text{PROBABILITIES} \\
\hline
1/6 & 6 & \frac{1}{6} \\
1/6 & NOT 6 & \frac{5}{6} \\
\hline
5/6 & 6 & \frac{5}{6} \\
5/6 & NOT 6 & \frac{1}{6} \\
\hline
6 & 6 & \frac{1}{6} \\
6 & NOT 6 & \frac{5}{6} \\
\hline
\text{total} & \frac{36}{36} & 1 \\
\end{array}
\]

(a) \( p(2 \text{ sixes}) = \frac{1}{36} \)

(b) \( p(1 \text{ six}) = \frac{5}{36} + \frac{5}{36} = \frac{10}{36} = \frac{5}{18} \)

(c) \( p(\text{no sixes}) = \frac{25}{36} \)

Note that these probabilities add up to 1. This will always be so when the probabilities are added from the outcome of the tree diagram. This is a very useful means of checking your working.
Example 2

A bag contains 4 red balls and 3 green balls. A ball is taken out at random, and then put back; a second ball is then taken from the bag. What is the probability that:

(a) both balls are the same colour,
(b) at least one of the balls is green,
(c) the balls are of different colours?

Solution

Use a tree diagram:

<table>
<thead>
<tr>
<th>1st Ball</th>
<th>2nd Ball</th>
<th>OUTCOMES</th>
<th>PROBABILITIES</th>
</tr>
</thead>
<tbody>
<tr>
<td>R</td>
<td>R</td>
<td>R R</td>
<td>( \frac{4}{7} \times \frac{4}{7} = \frac{16}{49} )</td>
</tr>
<tr>
<td>R</td>
<td>G</td>
<td>R G</td>
<td>( \frac{4}{7} \times \frac{3}{7} = \frac{12}{49} )</td>
</tr>
<tr>
<td>G</td>
<td>R</td>
<td>G R</td>
<td>( \frac{3}{7} \times \frac{4}{7} = \frac{12}{49} )</td>
</tr>
<tr>
<td>G</td>
<td>G</td>
<td>G G</td>
<td>( \frac{3}{7} \times \frac{3}{7} = \frac{9}{49} )</td>
</tr>
</tbody>
</table>

\[
\text{(a) } p(\text{both the same}) = p(\text{R R or G G}) = p(\text{R R}) + p(\text{G G}) = \frac{16}{49} + \frac{9}{49} = \frac{25}{49}
\]

\[
\text{(b) } p(\text{at least one green ball}) = p(\text{G G or G R or R G}) \text{ or } = 1 - p(\text{R R}) = 1 - \frac{16}{49} = \frac{33}{49}
\]
(c)  \( p(\text{both different colours}) = p(\text{R G or G R}) \)

\[ = p(\text{R G}) + p(\text{G R}) \]

\[ = \frac{12}{49} + \frac{12}{49} \]

\[ = \frac{24}{49} \]

Note: In probability questions of this type, ‘or’ means adding the probabilities.

**Example 3**

On her way to work, Sylvia drives through three sets of traffic lights. The probability of each set of lights being green is 0.3. What is the probability that they are all green?

**Solution**

\[ p(\text{all green}) = p(\text{1st green and 2nd green and 3rd green}) \]

\[ = p(\text{1st green}) \times p(\text{2nd green}) \times p(\text{3rd green}) \]

\[ = 0.3 \times 0.3 \times 0.3 \quad [\text{or } 0.3^3] \]

\[ = 0.027 \]

Note: In probability questions of this type, ‘and’ means multiplying the probabilities.

*Remember*  A tree diagram is drawn when it will help you to analyse a problem; so if it will help, draw one. On the other hand, if you are able to solve a problem without one (see Example 3 above), then do so.

**Example 4**

The diagram shows a model railway track. At each of the junctions P, Q and R, the probability of a train going straight ahead is \( \frac{2}{3} \) and the probability of it branching to the right is \( \frac{1}{3} \).
A train starts at point A.
(a) What is the probability that it reaches point C?
(b) What is the probability that it reaches point D?

Solution
(a) \( p(\text{right and straight and straight}) = \frac{1}{3} \times \frac{2}{3} \times \frac{2}{3} = \frac{4}{27} \)

(b) \( p(\text{(right and right) or (right and straight and right)}) = \frac{1}{3} \times \frac{1}{3} + \frac{1}{3} \times \frac{2}{3} \times \frac{1}{3} = \frac{1}{9} + \frac{2}{27} = \frac{5}{27} \)

Exercises
1. A bag contains 3 red balls and 2 blue balls. A ball is taken at random from the bag and then put back. A second ball is then taken out of the bag. What is the probability that:
   (a) both balls are red,
   (b) both balls are the same colour,
   (c) at least one of the balls is red?
2. Repeat question 1 for a bag with 7 red balls and 3 blue balls.
3. Two fair dice are rolled at the same time. Use a tree diagram to determine the probability of obtaining:
   (a) two even numbers,
   (b) at least one even number,
   (c) no even numbers.
4. Two fair dice are rolled at the same time. Use a tree diagram to determine the probability of obtaining:
   (a) two multiples of 3,
   (b) exactly one multiple of 3,
   (c) less than two multiples of 3.
5. A coin has been weighted, so that the probability of getting a head is $\frac{2}{5}$ and the probability of getting a tail is $\frac{3}{5}$; the coin is thrown twice. Determine the probability of obtaining:
   (a) 2 heads,  
   (b) no heads,  
   (c) at least one head.

6. The spinner shown in the diagram is spun twice. Use a tree diagram to determine the probability of obtaining:
   (a) 2 reds,  
   (b) at least one red,  
   (c) no reds.

7. The spinner in the diagram is spun twice. Determine the probability of obtaining:
   (a) at least one A,  
   (b) at least one B,  
   (c) two As,  
   (d) two Bs.

8. The spinner in question 6 is spun 3 times. Use a tree diagram to determine the probability of obtaining:
   (a) 3 reds,  
   (b) 2 reds,  
   (c) at least 1 red.

9. A bag contains 1 red ball, 2 green balls and 4 yellow balls. A ball is taken from the bag at random. The ball is then put back, and a second ball is taken at random from the bag.
   What is the probability that:
   (a) both balls are the same colour,  
   (b) no yellow balls are taken out,  
   (c) at least one yellow ball is taken out?

10. Each of 10 balls is marked with a different number from 1 to 10. One ball is taken at random and then replaced. A second ball is then taken at random. Determine the probability that:
    (a) both balls taken are marked with the number 5,  
    (b) both balls taken have even numbers,  
    (c) both balls taken have numbers which are multiples of 3,  
    (d) at least one of the balls taken has a number greater than 2.

11. On his way to work, Paul has to pass through 2 sets of traffic lights. The probability that the first set of lights is green is 0.5, and the probability that the second set of lights is green is 0.4.
   What is the probability that both sets of lights are green?
12. On her way to the theatre, Sheila passes through 3 sets of traffic lights. The probability that each set of lights is green is $\frac{1}{3}$.
   (a) What is the probability that none of the lights is green?
   (b) What is the probability that two sets of lights are green and the other set is not green?

13. The diagram shows a section of a railway track. At each of the junctions B, C, D and E, the probability of going straight on is $\frac{3}{4}$. The train starts at A.
   (a) What is the probability that it reaches P?
   (b) What is the probability that it reaches Q?

14. A rat leaves position R and starts walking towards B. If it reaches B it gets nothing, if it reaches A it gets food and if it reaches C it gets water.
   At each of the junctions W, X, Y and Z, the probability of going straight on is 0.6 and the probability of branching off is 0.4.
   (a) What is the probability that the rat gets food?
   (b) What is the probability that the rat gets water?
   (c) What is the probability that it gets nothing?

15. When two fair dice are thrown, what is the probability that the score on the second dice is higher than the score on the first dice?
10.5 Conditional Probability

In some situations where events are repeated, the probabilities will change after the first event. For example, consider a bag containing 8 red balls and 3 blue balls.

The probability that a ball taken at random is red is \( \frac{8}{11} \).

If a second ball is taken out without the first ball being replaced, then:

Either the first ball was red, so the probability that the second ball is red is \( \frac{7}{10} \), since there are 3 blue balls but only 7 red balls left.

Or the first ball was blue, so the probability that the second is red is \( \frac{8}{10} = \frac{4}{5} \), since there are 8 red balls but only 2 blue balls left.

Tree diagrams are very useful for this type of problems.

Example 1

A bag contains 7 yellow balls and 5 red balls. One ball is taken from the bag at random, and is not replaced. A second ball is then taken from the bag. Determine the probability that:

(a) both balls are red,  
(b) both balls are the same colour,  
(c) the balls are different colours,  
(d) at least one ball is yellow.

Solution

The tree diagram below shows the probabilities and outcomes:

<table>
<thead>
<tr>
<th>1st Ball</th>
<th>2nd Ball</th>
<th>OUTCOMES</th>
<th>PROBABILITIES</th>
</tr>
</thead>
<tbody>
<tr>
<td>Y</td>
<td>( \frac{6}{11} )</td>
<td>Y Y</td>
<td>( \frac{7}{12} \times \frac{6}{11} = \frac{42}{132} )</td>
</tr>
<tr>
<td>R</td>
<td>( \frac{5}{11} )</td>
<td>Y R</td>
<td>( \frac{7}{12} \times \frac{5}{11} = \frac{35}{132} )</td>
</tr>
<tr>
<td>R</td>
<td>( \frac{7}{11} )</td>
<td>R Y</td>
<td>( \frac{5}{12} \times \frac{7}{11} = \frac{35}{132} )</td>
</tr>
<tr>
<td>R</td>
<td>( \frac{4}{11} )</td>
<td>R R</td>
<td>( \frac{5}{12} \times \frac{4}{11} = \frac{20}{132} )</td>
</tr>
</tbody>
</table>

Total = \( \frac{132}{132} = 1 \)
(a) \( p(\text{both red}) = \frac{20}{132} = \frac{5}{33} \)

(b) \( p(\text{both the same colour}) = p(Y Y) + p(R R) \)
\[
= \frac{42}{132} + \frac{20}{132} = \frac{62}{132} = \frac{31}{66}
\]

(c) \( p(\text{different colours}) \)
\[
= 1 - p(\text{same colour}) \quad \text{or} \quad p(Y R) + p(R Y) = \frac{35}{132} + \frac{35}{132}
\]
\[
= 1 - \frac{31}{66} = \frac{70}{132}
\]
\[
= \frac{35}{66} = \frac{35}{66}
\]

(d) \( p(\text{at least one yellow}) \)
\[
= \frac{42}{132} + \frac{35}{132} + \frac{35}{132} \quad \text{or} \quad 1 - p(R R)
\]
\[
= \frac{112}{132} = 1 - \frac{20}{132}
\]
\[
= \frac{28}{33} = \frac{112}{132}
\]
\[
= \frac{28}{33}
\]

Example 2
There are 4 boys and 5 girls who are hoping to be selected for a school quiz team. Two of them are selected at random to be in the team.
Determine the probability that:
(a) 2 boys are chosen,
(b) at least 1 girl is chosen,
(c) 1 girl and 1 boy are chosen.
Solution

The tree diagram below shows the outcomes and the probabilities:

\[
\begin{array}{c|c|c|c}
\text{OUTCOMES} & \text{PROBABILITIES} \\
\hline
G G & \frac{5}{9} \times \frac{4}{8} = \frac{20}{72} = \frac{5}{18} \\
G B & \frac{5}{9} \times \frac{4}{8} = \frac{20}{72} = \frac{5}{18} \\
B G & \frac{4}{9} \times \frac{5}{8} = \frac{20}{72} = \frac{5}{18} \\
B B & \frac{4}{9} \times \frac{3}{8} = \frac{12}{72} = \frac{3}{18} \\
\hline
\text{total} & \frac{18}{18} = 1
\end{array}
\]

(a) \( p(2 \text{ boys}) = \frac{3}{18} = \frac{1}{6} \)

(b) \( p(\text{at least 1 girl}) = \frac{5}{18} + \frac{5}{18} + \frac{5}{18} = \frac{15}{18} = \frac{5}{6} \)

(c) \( p(1 \text{ boy and 1 girl}) = \frac{5}{18} + \frac{5}{18} = \frac{10}{18} = \frac{5}{9} \)

Note: The questions in Examples 1 and 2 could have been answered without the use of tree diagrams, but a tree diagram helps greatly with the analysis of the problem; the same is true for the next example.

Example 3

The probability that Ravi does his homework is \( \frac{1}{10} \) if he goes out with his friends and \( \frac{3}{5} \) of he does not go out with his friends. The probability that Ravi goes out with his friends is \( \frac{3}{4} \). What is the probability that Ravi does his homework?
Solution 1

\[ p((\text{goes out and does homework}) \text{ or } (\text{does not go out and does homework})) \]
\[ = p(\text{goes out}) \times p(\text{does homework}) + p(\text{does not go out}) \times p(\text{does homework}) \]
\[ = \frac{3}{4} \times \frac{1}{10} + \frac{1}{4} \times \frac{3}{5} \]
\[ = \frac{3}{40} + \frac{3}{20} \]
\[ = \frac{9}{40} \]

Solution 2

<table>
<thead>
<tr>
<th>OUTCOMES</th>
<th>PROBABILITIES</th>
</tr>
</thead>
<tbody>
<tr>
<td>O D</td>
<td>( \frac{3}{4} \times \frac{1}{10} = \frac{3}{40} )</td>
</tr>
<tr>
<td>O' D'</td>
<td>( \frac{3}{4} \times \frac{9}{10} = \frac{27}{40} )</td>
</tr>
<tr>
<td>O' D</td>
<td>( \frac{1}{4} \times \frac{3}{5} = \frac{3}{20} )</td>
</tr>
<tr>
<td>O' D'</td>
<td>( \frac{1}{4} \times \frac{2}{5} = \frac{2}{20} = \frac{1}{10} )</td>
</tr>
</tbody>
</table>

Note: O' means does not go out, and D' means does not do homework.

\[ p(\text{does homework}) = \frac{3}{40} + \frac{3}{20} \]
\[ = \frac{9}{40} \]

Exercises

1. A bag contains 3 pink balls and 2 blue balls. One ball is taken out at random and not replaced. A second ball is then taken out. Determine the probability that:
   (a) both balls are pink,
   (b) both balls are the same colour,
   (c) at least one ball is blue.
2. In Tim's drawer there are 6 black socks and 5 white socks. He takes out two socks at random. What is the probability that he has taken two socks of the same colour?

3. In a tennis club there are 5 boys and 3 girls in a training squad. Two are chosen at random to represent the club. Determine the probability that they are:
   (a) both boys,
   (b) both girls,
   (c) a boy and a girl.

4. Tara has five 10p coins and four 20p coins in her purse. She takes out two coins at random. What is the probability that she takes out at least 30p?

5. There are 8 footballs in a store cupboard; one is yellow and the others are white. A pupil takes 2 footballs out of the cupboard at random. What is the probability that one of them is the yellow ball?

6. The probability of Jeremy passing a maths exam is \( \frac{2}{3} \) if he revises and \( \frac{1}{3} \) if he does not revise. The probability that he revises is \( \frac{1}{4} \). What is the probability of Jeremy passing the maths exam?

7. The probability of Jenny getting to work on time is 0.8 if she gets up before 7 a.m. and 0.4 if she does not get up before 7 a.m. The probability that Jenny gets up before 7 a.m. is 0.7. What is the probability that Jenny is late for work?

8. Ian is an inept mountaineer who tends to fall from rock faces. The probability that he falls is 0.2 if the weather is dry but rises to 0.5 if it is wet. The probability of wet weather is 0.3. Determine the probability that Ian falls.

9. A bag contains 7 blue counters, 5 green counters, 2 black counters and 1 white counter. 3 counters are taken at random from the bag, without replacement. What is the probability that they are all the same colour?

10. Peter and Jane play a game in which they each in turn take a counter at random from a bag containing 7 red counters and 3 yellow counters. The winner is the first to get a red counter. Jane goes first. By drawing a tree diagram, determine the probability that Peter wins the game.
11 Angles, Bearings and Maps

11.1 Angle Measures

In this section we review measuring angles, and the different types of angles.

<table>
<thead>
<tr>
<th>Acute angle</th>
<th>Right angle</th>
<th>Obtuse angle</th>
<th>Straight line</th>
<th>Reflex angle</th>
<th>Complete turn</th>
</tr>
</thead>
<tbody>
<tr>
<td>less than 90°</td>
<td>= 90°</td>
<td>between 90° and 180°</td>
<td>= 180°</td>
<td>greater than 180°</td>
<td>= 360°</td>
</tr>
</tbody>
</table>

Example 1
Measure the angle in the diagram.

Solution
Using a protractor, the angle can be measured as 35°.

Example 2
State whether each of the angles below is acute, obtuse or reflex.

A / B / C / D / E
Solution

A  *Obtuse* as it is between $90^\circ$ and $180^\circ$.
B  *Reflex* as it is greater than $180^\circ$.
C  *Acute* as it is less than $90^\circ$.
D  *Reflex* as it is greater than $180^\circ$.
E  *Obtuse* as it is between $90^\circ$ and $180^\circ$.

Exercises

1. Measure the following angles:
   (a)  
   (b)  
   (c)  
   (d)  

2. Measure the following angles:
   (a)  
   (b)  
3. State whether each of the following angles is *acute*, *obtuse* or *reflex*.

(a)  (b)  (c)  
(d)  (e)  (f)
4. (a) Measure the angles in the triangle below:

(b) What is the sum of the three angles?

5. (a) Measure the angles in the quadrilateral opposite:

(b) What is the sum of the four angles?

6. (a) Without using a protractor, try to draw an angle of 45°.

(b) Measure your angle to see how accurate you were.

7. (a) Draw the angle shown in the diagram.

(b) Measure the acute angle that you also draw.

(c) Check that the two angles add up to 360°.

8. (a) Measure the three angles marked in the diagram.

(b) Check that they add up to 360°.
9. (a) Measure the two angles in the diagram.
   (b) Check that they add up to 180 °.

10. (a) Without using a protractor, try to draw an angle of 300 °.
    (b) Check your answer by measuring the angle with a protractor.

11.2 Parallel and Intersecting Lines

When a line intersects (or crosses) a pair of parallel lines, there are some simple rules that can be used to calculate unknown angles.

The arrows on the lines indicate that they are parallel.

\[
\begin{align*}
  a &= b \quad \text{(and } c = d, \text{ and } e = f) \quad \text{These are called vertically opposite angles.} \\
  a &= c \quad \text{(and } b = d) \quad \text{These are called corresponding angles.} \\
  b &= c \quad \text{These are called alternate angles.} \\
  a + e &= 180 °, \text{ because adjacent angles on a straight line add up to 180 °.} \\
  \quad \text{These are called supplementary angles.} \\
  c + e &= 180 ° \quad \text{(allied or supplementary angles)}
\end{align*}
\]
Example 1
In the diagram opposite, find the unknown angles if \( a = 150 ^\circ \).

**Solution**

To find \( b \):

\[
\begin{align*}
  a + b &= 180 ^\circ \quad \text{(angles on a straight line, supplementary angles)} \\
  150 ^\circ + b &= 180 ^\circ \\
  b &= 30 ^\circ 
\end{align*}
\]

To find \( c \):

\[
\begin{align*}
  c &= b \quad \text{(vertically opposite angles or angles on a straight line)} \\
  c &= 30 ^\circ 
\end{align*}
\]

To find \( d \):

\[
\begin{align*}
  d &= a \quad \text{(corresponding angles)} \\
  d &= 150 ^\circ 
\end{align*}
\]

To find \( e \):

\[
\begin{align*}
  e &= c \quad \text{(corresponding angles)} \\
  e &= 30 ^\circ 
\end{align*}
\]

Example 2
Find the size of the unknown angles in the parallelogram shown in this diagram:

**Solution**

To find \( a \):

\[
\begin{align*}
  a + 70 ^\circ &= 180 ^\circ \quad \text{(allied or supplementary angles)} \\
  a &= 110 ^\circ 
\end{align*}
\]

To find \( b \):

\[
\begin{align*}
  b + a &= 180 ^\circ \quad \text{(allied or supplementary angles)} \\
  b + 110 ^\circ &= 180 ^\circ \\
  b &= 70 ^\circ 
\end{align*}
\]
To find $c$:

$$c + 70^\circ = 180^\circ \quad \text{(allied or supplementary angles)}$$

$$c = 110^\circ$$

or

$$c = 360^\circ - (a + b + 70^\circ) \quad \text{(angle sum of a quadrilateral)}$$

$$= 360^\circ - 250^\circ$$

$$= 110^\circ$$

or

$$c = a \quad \text{(opposite angles of a parallelogram are equal)}$$

Exercises

1. Which angles in the diagram are the same size as:
   (a) $a,$
   (b) $b$?

2. Find the size of each of the angles marked with letters in the diagrams below, giving reasons for your answers:
   (a)
   (b)
   (c)
   (d)

3. Find the size of the three unknown angles in the parallelogram opposite:
4. One angle in a parallelogram measures $36^\circ$. What is the size of each of the other three angles?

5. One angle in a rhombus measures $133^\circ$. What is the size of each of the other three angles?

6. Find the sizes of the unknown angles marked with letters in the diagram:

7. (a) In the diagram opposite, find the sizes of the angles marked in the triangle. Give reasons for your answers.

(b) What special name is given to the triangle in the diagram?

8. The diagram shows a bicycle frame. Find the sizes of the unknown angles $a$, $b$ and $c$.

9. BCDE is a trapezium.

(a) Find the sizes of all the unknown angles, giving reasons for your answers.

(b) What is the special name given to this type of trapezium?
11.3 Bearings

Bearings are a measure of direction, with north taken as a reference. If you are travelling north, your bearing is 000°.

If you walk from O in the direction shown in the diagram, you are walking on a bearing of 110°.

Bearings are always measured clockwise from north, and are given as three figures, for example:

- Bearing 060°
- Bearing 240°
- Bearing 330°

Example 1

On what bearing is a ship sailing if it is heading:

(a) E, (b) S, (c) W, (d) SE, (e) NW?

Solution

(a) E

Bearing is 090°.

(b) S

Bearing is 180°.

(c) W

Bearing is 270°.
Example 2

A ship sails from A to B on a bearing of $060^\circ$. On what bearing must it sail if it is to return from B to A?

Solution

The diagram shows the journey from A to B.

Extending the line of the journey allows an angle of $60^\circ$ to be marked at B.

Bearing of A from B = $60^\circ + 180^\circ$

= $240^\circ$

and this is called a back bearing or a reciprocal bearing.

Exercises

1. What angle do you turn through if you turn clockwise from:
   (a) N to S,
   (b) E to W,
   (c) N to NE,
   (d) N to SW,
   (e) W to NW?

2. Copy and complete the table:

<table>
<thead>
<tr>
<th>Direction</th>
<th>Bearing</th>
</tr>
</thead>
<tbody>
<tr>
<td>N</td>
<td></td>
</tr>
<tr>
<td>NE</td>
<td></td>
</tr>
<tr>
<td>W</td>
<td></td>
</tr>
<tr>
<td>SW</td>
<td></td>
</tr>
</tbody>
</table>
3. The map of an island is shown below:

What is the bearing from the tower, of each place shown on the map?

4. The diagram shows the positions of two ships, A and B.
   (a) What is the bearing of ship A from ship B?
   (b) What is the bearing of ship B from ship A?

5. The diagram shows 3 places, A, B and C. Find the bearing of:
   (a) A from C,
   (b) B from A,
   (c) C from B,
   (d) B from C.

6. An aeroplane flies from Newquay to Birmingham on a bearing of 044°. On what bearing should the pilot fly, to return to Newquay from Birmingham?
7. On four separate occasions, a plane leaves Exeter airport to fly to a different destination. The bearings of these destinations from Exeter airport are given below.

<table>
<thead>
<tr>
<th>Destination</th>
<th>Bearing</th>
</tr>
</thead>
<tbody>
<tr>
<td>London</td>
<td>077 °</td>
</tr>
<tr>
<td>Glasgow</td>
<td>356 °</td>
</tr>
<tr>
<td>Leeds</td>
<td>036 °</td>
</tr>
<tr>
<td>Guernsey</td>
<td>162 °</td>
</tr>
</tbody>
</table>

Copy and complete the diagram to show the direction in which the plane flies to each destination.

8. A ship sails NW from a port to take supplies to an oil rig. On what bearing must it sail to return from the oil rig to the port?

9. If A is north of B, C is southeast of B and on a bearing of 160 ° from A, find the bearing of:
   (a) A from B,
   (b) A from C,
   (c) C from B,
   (d) B from C.

10. If A is on a bearing of 300 ° from O, O is NE of B, and the bearing of B from A is 210 °, find the bearing of:
    (a) A from B,
    (b) O from A,
    (c) O from B.
11.4 Scale Drawings

Using bearings, scale drawings can be constructed to solve problems.

Example 1

A ship sails 20 km NE, then 18 km S, and then stops.
(a) How far is it from its starting point when it stops?
(b) On what bearing must it sail to return to its starting point?

Solution

The path of the ship can be drawn using a scale of 1 cm for every 2 km, as shown in the diagram.

(a) The distance BO can be measured on the diagram as 7.3 cm which represents an actual distance of 14.6 km.

(b) The bearing of O from B can be measured as 285°.
Note: Remember to always put the scale on the diagram.

Example 2

A man walks 750 m on a bearing of 030°. He then walks on a bearing of 315° until he is due north of his starting point, and stops.
(a) How far does he walk on the bearing of 315°?
(b) How far is he from his starting point when he stops?
Solution

A scale drawing can be produced, using a scale of 1 cm to 100 m.

(a) The distance $AB$ can be measured as 5.4 cm, which represents an actual distance of 530 m.

(b) The distance $OB$ can be measured as 10.2 cm, representing an actual distance of 1020 m.

Exercises

1. A girl walks 80 m north and then 200 m east.
   (a) How far is she from her starting position?
   (b) On what bearing should she walk to get back to her starting position?

2. Andrew walks 300 m NW and then walks 500 m south and then stops.
   (a) How far is he from his starting position when he stops?
   (b) On what bearing could he have walked to go directly from his starting position to where he stopped?
3. An aeroplane flies 400 km on a bearing of $055^\circ$. It then flies on a bearing of $300^\circ$, until it is due north of its starting position. How far is the aeroplane from its starting position?

4. A captain wants to sail his ship from port A to port B, but the journey cannot be made directly. Port B is 50 km north of A.

The ship sails 20 km on a bearing of $075^\circ$.

It then sails 20 km on a bearing of $335^\circ$ and then drops anchor.

(a) How far is the ship from port B when it drops anchor?

(b) On what bearing should the captain sail the ship to arrive at port B?

5. Julie intended to walk 200 m on a bearing of $240^\circ$. Her compass did not work properly, so she actually walked 200 m on a bearing of $225^\circ$. What distance and on what bearing should she walk to get to the place she intended to reach?

6. A hot air balloon is blown 5 km NW. The wind then changes direction and the balloon is blown a further 6 km on a bearing of $300^\circ$ before landing. How far is the balloon from its starting point when it lands?

7. Robin and Jane set off walking at the same time. When they start, Robin is 6 km NW of Jane. Jane walks 3 km on a bearing of $350^\circ$ and Robin walks 4 km on a bearing of $020^\circ$. How far apart are they now?

8. An aeroplane flies 200 km on a bearing of $335^\circ$. It then flies 100 km on a bearing of $170^\circ$ and 400 km on $280^\circ$, and then lands.

(a) How far is the aeroplane from its starting point when it lands?

(b) On what bearing could it have flown to complete its journey directly?

9. Brian is sailing on a bearing of $135^\circ$. After his boat has travelled 20 km, he realises that he is 1 km north of the port that he wanted to reach.

(a) On what bearing should he have sailed?

(b) How far from his starting point is the port that he wanted to reach?

10. A pilot knows that to fly to another airport he needs to fly 500 km on a bearing of $200^\circ$. When he has flown 400 km, he realises that he is 150 km from the airport.
(a) On what bearing has the pilot been flying?
(b) On what bearing should he fly to reach the airport?
(Note that there are two answers.)

11. Four planes take off from Exeter airport, each one flying on a different bearing to another UK airport. The bearings and the distances from Exeter to these airports are given in the table below.

<table>
<thead>
<tr>
<th>Destination</th>
<th>Bearing</th>
<th>Distance</th>
</tr>
</thead>
<tbody>
<tr>
<td>London</td>
<td>077 °</td>
<td>255 km</td>
</tr>
<tr>
<td>Glasgow</td>
<td>356 °</td>
<td>575 km</td>
</tr>
<tr>
<td>Leeds</td>
<td>036 °</td>
<td>390 km</td>
</tr>
<tr>
<td>Guernsey</td>
<td>162 °</td>
<td>150 km</td>
</tr>
</tbody>
</table>

Using a scale of 1 cm to represent 50 km, draw a map showing the positions of the five airports.
12 Formulae

12.1 Substitution 1

In this section we practise substituting numbers for the letters in a formula: in other words, we replace the letters in formulae with their numerical values.

Example 1

If $a = 6$, $b = 3$ and $c = 7$, calculate the value of:

(a) $a + b$  
(b) $a - b$  
(c) $a + c$  
(d) $c + b - a$

Solution

(a) $a + b = 6 + 3 = 9$
(b) $a - b = 6 - 3 = 3$
(c) $a + c = 6 + 7 = 13$
(d) $c + b - a = 7 + 3 - 6 = 4$

Reminders

We write: $2 \times a$ as $2a$

$a \times b$ as $ab$

$a \div 4$ as $\frac{a}{4}$

and $a \div b$ as $\frac{a}{b}$

Example 2

If $p = 6$, $q = 12$, $r = 4$ and $s = 3$, evaluate:

(a) $rs$  
(b) $4p$  
(c) $2r + 3s$
(d) $\frac{s}{3}$  
(e) $\frac{q}{s}$
Solution

(a) \( r s = r \times s \)
\[ = 4 \times 3 \]
\[ = 12 \]

(b) \( 4 p = 4 \times p \)
\[ = 4 \times 6 \]
\[ = 24 \]

(c) \( 2 r + 3 s = 2 \times r + 3 \times s \)
\[ = 2 \times 4 + 3 \times 3 \]
\[ = 8 + 9 \]
\[ = 17 \]

(d) \( \frac{s}{3} = \frac{3}{3} \)
\[ = 1 \]

(e) \( \frac{q}{s} = \frac{12}{3} \)
\[ = 4 \]

Example 3

The perimeter of the trapezium shown is given by the formula
\[ p = a + b + c + d \]

Calculate the perimeter if \( a = 4, \quad b = 6, \quad c = 8 \text{ and } d = 4. \)

Solution

Perimeter \( p = a + b + c + d \)
\[ = 4 + 6 + 8 + 4 \]
\[ = 22 \]
Exercises

1. Calculate the values of the following expressions, if \( x = 2, \ y = 5 \) and \( z = 9 \):
   (a) \( x + y \)  
   (b) \( x + z \)  
   (c) \( y + z \)  
   (d) \( z - y \)  
   (e) \( y - x \)  
   (f) \( z - x \)  
   (g) \( x + y + z \)  
   (h) \( z + y - x \)  
   (i) \( z - y + x \)

2. If \( p = 7, \ q = 2 \) and \( r = 3 \), evaluate the following expressions:
   (a) \( 2p \)  
   (b) \( 4r \)  
   (c) \( 5q \)  
   (d) \( 5p \)  
   (e) \( 6r \)  
   (f) \( 2q \)  
   (g) \( 3p \)  
   (h) \( 10p \)  
   (i) \( 8r \)

3. If \( i = 6, \ j = 7, \ k = 3 \) and \( l = 4 \), determine the values of the following expressions:
   (a) \( 2i + 3k \)  
   (b) \( 2l + 3i \)  
   (c) \( 2j + 5l \)  
   (d) \( 5j + 6k \)  
   (e) \( 4i + 3l \)  
   (f) \( 10j + 6l \)  
   (g) \( 3i - j \)  
   (h) \( 4k - i \)  
   (i) \( 6l - 2k \)  
   (j) \( 3i - 2j \)  
   (k) \( 7k - 2i \)  
   (l) \( 8l - 5k \)

4. If \( s = 10, \ t = 12, \ u = 15 \) and \( v = 20 \), evaluate the following expressions:
   (a) \( \frac{s}{2} \)  
   (b) \( \frac{t}{3} \)  
   (c) \( \frac{u}{5} \)  
   (d) \( \frac{v}{10} \)  
   (e) \( \frac{v}{2} \)  
   (f) \( \frac{u}{3} \)  
   (g) \( \frac{t}{6} \)  
   (h) \( \frac{s}{10} \)  
   (i) \( \frac{u}{1} \)

5. If \( e = 10, \ f = 20, \ g = 5 \) and \( h = 4 \), determine the values of the following expressions:
   (a) \( eg \)  
   (b) \( gh \)  
   (c) \( ef \)  
   (d) \( eh \)  
   (e) \( \frac{e}{g} \)  
   (f) \( \frac{f}{h} \)  
   (g) \( \frac{f}{g} \)  
   (h) \( \frac{f}{e} \)  
   (i) \( \frac{e}{g} \)
6. In a sweet shop you can buy packets of mints for 20p each and bars of chocolate for 30p each. The total cost of \( m \) packets of mints and \( c \) bars of chocolate is given by the formula

\[
T = 20m + 30c
\]

Use this formula to calculate the total cost if:

(a) \( m = 2 \) and \( c = 1 \)  
(b) \( m = 8 \) and \( c = 0 \)  
(c) \( m = 3 \) and \( c = 3 \)  
(d) \( m = 5 \) and \( c = 4 \)  
(e) \( m = 1 \) and \( c = 10 \)  
(f) \( m = 2 \) and \( c = 3 \)

7. The perimeter of the rectangle shown is given by the formula

\[
p = 2l + 2w
\]

Calculate the perimeter of rectangles for which:

(a) \( l = 2, \ w = 1 \)  
(b) \( l = 8, \ w = 2 \)  
(c) \( l = 10, \ w = 9 \)  
(d) \( l = 10, \ w = 3 \)

8. The perimeter of the triangle shown is given by the formula

\[
p = x + y + z
\]

Determine \( p \) if:

(a) \( x = 4, \ y = 8 \) and \( z = 6 \)  
(b) \( x = 2, \ y = 3 \) and \( z = 4 \)  
(c) \( x = 10, \ y = 17 \) and \( z = 20 \)  
(d) \( x = 9, \ y = 14 \) and \( z = 15 \)

9. The cost of entry to a leisure park for an adult is £5 and for a child is £4. The total cost in pounds for \( a \) adults and \( c \) children is given by the formula

\[
T = 5a + 4c
\]

Calculate the cost if:

(a) \( a = 2 \) and \( c = 4 \)  
(b) \( a = 7 \) and \( c = 1 \)  
(c) \( a = 1 \) and \( c = 5 \)  
(d) \( a = 2 \) and \( c = 3 \)  
(e) \( a = 3 \) and \( c = 8 \)  
(f) \( a = 10 \) and \( c = 30 \)
10. The time, $T$ hours, taken to drive $D$ kilometres along a motorway at a speed of $S$ kilometres per hour is calculated using the formula 

$$T = \frac{D}{S}$$

Calculate the time taken if:

(a) $D = 200$ and $S = 100$
(b) $D = 160$ and $S = 80$
(c) $D = 360$ and $S = 60$
(d) $D = 5$ and $S = 10$

12.2 Substitution 2

In this section we look at substituting positive and negative values; we also look at more complex equations.

**Reminder**

| B | B rackets |
| O | BODMAS | can be used to remember the order in which to carry out operations |
| D | D ivision |
| M | M ultiplcation |
| A | A ddition |
| S | S ubtraction |

**Example 1**

If $p = 2(x + y)$, calculate the value of $p$ when $x = 3$ and $y = 5$.

**Solution**

\[
p = 2(x + y) \\
= 2(3 + 5) \\
= 2 \times 8 \\
= 16
\]
Example 2

A formula states that $Q = uv - \frac{v}{4}$.

Determine the value of $Q$ if $u = 8$ and $v = 12$.

Solution

$$Q = uv - \frac{v}{4} = 8 \times 12 - \frac{12}{4} = 96 - 3 = 93$$

**Reminders on adding and subtracting negative numbers**

To *add a positive number*, *move to the right* on a number line.

To *add a negative number*, *move to the left* on a number line.

To *subtract a positive number*, *move to the left* on a number line.

To *subtract a negative number*, *move to the right* on a number line.

For example:

$(-3) + 8 = +5$  (more usually written as $5$)

$7 + (-4) = 3$

$(-2) - 6 = -8$

and $(-3) - (-9) = 6$
Reminders on multiplying and dividing negative numbers

The table shows what happens to the sign of the answer when positive and negative numbers are multiplied or divided.

<table>
<thead>
<tr>
<th>× or ÷</th>
<th>+</th>
<th>−</th>
</tr>
</thead>
<tbody>
<tr>
<td>+</td>
<td>+</td>
<td>−</td>
</tr>
<tr>
<td>−</td>
<td>−</td>
<td>+</td>
</tr>
</tbody>
</table>

For example:  
\[3 \times (-7) = -21\]  
\[(-8) \times (-11) = 88\]  
\[12 \div (-6) = -2\]  
and \[(-15) \div (-3) = 5\]

Example 3

If \[p = 9, \; q = -4, \; r = 2\] and \[s = -7\], determine the values of the following expressions:

(a) \[p + q\]  
(b) \[p - q\]  
(c) \[q - r + s\]  
(d) \[ps\]  
(e) \[\frac{q}{r}\]  
(f) \[\frac{r}{q}\]  

Solution

(a) \[p + q = 9 + (-4)\]  
    \[= 5\]
(b) \[p - q = 9 - (-4)\]  
    \[= 13\]
(c) \[q - r + s = (-4) - 2 + (-7)\]  
    \[= -13\]
(d) \[ps = 9 \times (-7)\]  
    \[= -63\]
(e) \[ \frac{q}{r} = \frac{-4}{2} = -2 \]
(f) \[ \frac{r}{q} = \frac{2}{-4} = \frac{-1}{2} \]

Example 4
If \( a = 6 \), \( b = 5 \) and \( c = -2 \), determine the value of:

(a) \( abc \)  
(b) \( a(b + c) \)  
(c) \( ab - bc \)  
(d) \( \sqrt{a(b + 1)} \)  
(e) \( \frac{ab}{2} + bc^2 \)

Solution
(a) \( abc = 6 \times 5 \times (-2) = -60 \)

(b) \( a(b + c) = 6(5 + (-2)) = 6 \times 3 = 18 \)

(c) \( ab - bc = 6 \times 5 - 5 \times (-2) = 30 - (-10) = 30 + 10 = 40 \)

(d) \( \sqrt{a(b + 1)} = \sqrt{6 \times (5 + 1)} = \sqrt{6 \times 6} = \sqrt{36} = 6 \)
(c) \( \frac{ab}{2} + bc^2 = \frac{6 \times 5}{2} + 5 \times (-2)^2 \)
\[= \frac{30}{2} + 5 \times 4\]
\[= 15 + 20\]
\[= 35\]

Exercises

1. Calculate:
   (a) \(6 + (-2)\)  
   (b) \((-3) + 5\)  
   (c) \((-4) + (-2)\)  
   (d) \(2 - 4\)  
   (e) \(3 - (-2)\)  
   (f) \((-7) - (-4)\)  
   (g) \(2 \times (-6)\)  
   (h) \((-10) \times 5\)  
   (i) \((-12) \times (-4)\)  
   (j) \((-8) \div 4\)  
   (k) \(14 \div (-7)\)  
   (l) \((-25) \div (-5)\)  
   (m) \((-3)^2\)  
   (n) \((-5)^2 \times (-2)\)  
   (o) \((4 \times 5) + (-2)\)  
   (p) \((-3) \times (-4) \div 6\)  
   (q) \((-3) \times (-8) + (-7)\)  
   (r) \(\frac{(-6) \times (-4)}{(-12)}\)  
   (s) \(\frac{(-10)^2}{4}\)  
   (t) \((-3) \times (-5) \times (-9)\)  
   (u) \((-5)^2 + (-6)^2\)  

2. If \(a = 6,\ b = 3\) and \(c = 7\), calculate:
   (a) \(ab\)  
   (b) \(b + c\)  
   (c) \(c - a\)  
   (d) \(4b + 6c\)  
   (e) \(4c - 2b\)  
   (f) \(6a - 2c\)  
   (g) \(abc\)  
   (h) \(ab - bc\)  
   (i) \(2bc + ac\)  
   (j) \(b^2\)  
   (k) \(a^2 - b^2\)  
   (l) \(a^2 + b^2 - c^2\)  

3. If \(a = 2,\ b = -4\) and \(c = -5\), evaluate:
   (a) \(a^2 + b^2\)  
   (b) \(ab\)  
   (c) \(bc\)  
   (d) \(a - b\)  
   (e) \(c - b\)  
   (f) \(3a + 2c\)  
   (g) \(2a - 4c\)  
   (h) \(3a + 2b\)  
   (i) \(ab - ac\)
4. Calculate $\sqrt{a + b}$ when $a = 15$, $b = 2$ and $c = -3$.

5. A formula for the perimeter of a triangle is $p = x + y + z$, where $x$, $y$ and $z$ are the lengths of the three sides. Calculate the value of $p$ when $x = 1\frac{1}{2}$ cm, $y = 2\frac{1}{2}$ cm and $z = 3\frac{1}{2}$ cm.

6. The area of a trapezium is given by the formula

\[ A = \frac{1}{2}(a + b)h \]

Calculate the area of the trapezium for which $a = 3$ cm, $b = 3.6$ cm and $h = 2.2$ cm.

7. The length of one side of a right-angled triangle is given by the following formula:

\[ l = \sqrt{h^2 - x^2} \]

Calculate the length $l$, if $h = 13$ cm and $x = 12$ cm.

8. The following formula can be used to convert temperatures from degrees Celsius ($C$) to degrees Fahrenheit ($F$):

\[ F = 32 + \frac{9C}{5} \]

Calculate the value of $F$, if:

(a) $C = 100$  
(b) $C = 20$  
(c) $C = -10$  
(d) $C = -20$

9. A formula states that

\[ s = \frac{1}{2}(u + v)t \]

Calculate the value of $s$, if:

(a) $u = 3$, $v = 6$ and $t = 10$  
(b) $u = -2$, $v = 4$ and $t = 2$  
(c) $u = -10$, $v = -6$ and $t = 3$  
(d) $u = -20$, $v = -40$ and $t = 3$
10. A formula states that:

\[ f = \frac{u + v}{uv} \]

(i) Calculate the value of \( f \), if:

(a) \( u = 10 \) and \( v = 5 \)  
(b) \( u = 2 \) and \( v = 5 \)  
(c) \( u = 20 \) and \( v = 10 \)

(ii) Show that you obtain the same value in each case using the formula

\[ f = \frac{1}{u} + \frac{1}{v} \]

Why does this happen?

11. Alan knows that \( x = 2 \), \( y = -6 \) and \( z = -4 \). He calculates that \( Q = \frac{1}{4} \). Which of the formulae below could he have used?

- Formula A: \( Q = \frac{xy + yz}{xyz} \)
- Formula B: \( Q = \frac{yz - xy}{xyz} \)
- Formula C: \( Q = \frac{1}{x} - \frac{1}{z} \)
- Formula D: \( Q = \frac{1}{z} + \frac{1}{x} \)
- Formula E: \( Q = \frac{1}{x} + \frac{1}{y} \)

12.3 Linear Equations 1

In this section we revise the solution of simple equations.

**Reminder**

Whatever you do to one side of an equation you must do to the other side: it is like keeping a set of scales balanced.

It is conventional to give the solution to an equation with the unknown value on the left hand side, and its value on the right hand side, e.g. \( x = 4 \) not \( 4 = x \).
Example 1
Solve the following equations:

(a) \( x + 3 = 7 \)
(b) \( 13 = 5 + a \)
(c) \( y - 3 = 8 \)
(d) \( 11 = p - 4 \)

Solution

(a) \( x + 3 = 7 \)
   
   \[ \text{[Subtract 3 from both sides]} \quad x = 7 - 3 \]
   \[ x = 4 \]

(b) \( 13 = 5 + a \)
   
   \[ \text{[Subtract 5 from both sides]} \quad 13 - 5 = a \]
   \[ 8 = a \]
   \[ a = 8 \]

(c) \( y - 3 = 8 \)
   
   \[ \text{[Add 3 to both sides]} \quad y = 8 + 3 \]
   \[ y = 11 \]

(d) \( 11 = p - 4 \)
   
   \[ \text{[Add 4 to both sides]} \quad 11 + 4 = p \]
   \[ 15 = p \]
   \[ p = 15 \]

Example 2
Solve the following equations:

(a) \( 6x = 24 \) \hspace{1cm} (b) \( 15 = 3t \) \hspace{1cm} (c) \( \frac{w}{2} = 9 \)

Solution

(a) \( 6x = 24 \)
   
   \[ \text{[Divide both sides by 6]} \quad x = \frac{24}{6} \]
   \[ x = 4 \]
(b) \[ 15 = 3t \]
\[
\frac{15}{3} = t \\
5 = t \\
t = 5
\]

(c) \[ \frac{w}{2} = 9 \]
\[
\text{[Multiply both sides by 2]} \\
w = 9 \times 2 \\
w = 18
\]

Example 3

The length of the rectangle shown is 4 metres, and its width is \( x \) metres.

The area of the rectangle is 8 m\(^2\).

(a) Use this information to write down an equation involving \( x \).

(b) Solve the equation to determine the value of \( x \).

(c) What is the width of the rectangle in cm?

Solution

(a) The area of the rectangle is \((4 \times x)\) m\(^2\), and we are told that this equals 8 m\(^2\).

So the equation is \( 4 \times x = 8 \), which we write as \( 4x = 8 \).

(b) \[ 4x = 8 \]

\[
\text{[Divide both sides by 4]} \\
x = \frac{8}{4} \\
x = 2 \text{ m}
\]

(c) The width of the rectangle is 200 cm.
Exercises

1. Solve the following equations:
   (a) \( x + 5 = 9 \)  
   (b) \( x + 11 = 12 \)  
   (c) \( 7 + x = 9 \)  
   (d) \( x + 2 = 17 \)  
   (e) \( 14 = x + 6 \)  
   (f) \( x - 2 = 10 \)  
   (g) \( x - 6 = 5 \)  
   (h) \( 2 = x - 9 \)  
   (i) \( x + 3 = 0 \)  
   (j) \( x - 7 = 7 \)  
   (k) \( x + 12 = 7 \)  
   (l) \( x - 6 = -10 \)  

2. Solve the following equations:
   (a) \( 2x = 12 \)  
   (b) \( 3x = 18 \)  
   (c) \( 5x = 20 \)  
   (d) \( 7x = 21 \)  
   (e) \( 36 = 9x \)  
   (f) \( 5x = 0 \)  
   (g) \( 80 = 10x \)  
   (h) \( \frac{x}{2} = 5 \)  
   (i) \( \frac{x}{3} = 6 \)  
   (j) \( 9 = \frac{x}{4} \)  
   (k) \( \frac{x}{2} = 22 \)  
   (l) \( \frac{x}{7} = 4 \)  
   (m) \( 4x = 2 \)  
   (n) \( \frac{x}{2} = 6 \)  
   (o) \( \frac{x}{10} = 0 \)  

3. Solve the following equations:
   (a) \( x + 7 = 9 \)  
   (b) \( x - 6 = 8 \)  
   (c) \( 3x = 33 \)  
   (d) \( \frac{x}{5} = 2 \)  
   (e) \( x + 2 = 13 \)  
   (f) \( 5x = 35 \)  
   (g) \( 4 + x = 15 \)  
   (h) \( 7 = y - 9 \)  
   (i) \( 42 = 6p \)  
   (j) \( 90 = \frac{q}{9} \)  
   (k) \( 5r = -10 \)  
   (l) \( -4 = \frac{s}{8} \)  

4. The area of the rectangle shown is 18 cm². 
   (a) Write down an equation involving \( x \).  
   (b) Solve your equation.  
   (c) Write down the width of the rectangle.  

5. The perimeter of the triangle shown is 17 cm. 
   (a) Write down an equation and solve it for \( x \).  
   (b) Write down the length of the side marked \( x \).
12.4 Linear Equations 2

In this section we solve linear equations where more than one step is needed to reach the solution. There are no simple rules here, since methods of solution vary from one equation to another.

Example 1
Solve the equation,

\[5x + 2 = 17\]

Solution
The first step is to subtract 2 from both sides, giving

\[5x = 15\]

Secondly, divide both sides by 5, to give the solution

\[x = 3\]

Example 2
Solve the equation,

\[4x - 7 = 17\]

Solution
[Add 7 to both sides] \[4x = 24\]
[Divide both sides by 4] \[x = 6\]

Example 3
Solve the equation,

\[5(7 + 2x) = 65\]

Solution

\[EITHER\]

[Multiply out the brackets] \[35 + 10x = 65\]
[Subtract 35 from both sides] \[10x = 30\]
[Divide both sides by 3] \[x = 3\]

\[OR\]

[Divide both sides by 5] \[7 + 2x = 13\]
[Subtract 7 from both sides] \[2x = 6\]
[Divide both sides by 2] \[x = 3\]
Example 4
Solve the equation,

\[ 6x - 2 = 4x + 8 \]

**Solution**

[Subtract 4\(x\) from both sides] \[2x - 2 = 8\]

[Add 2 to both sides] \[2x = 10\]

[Divide both sides by 2] \[x = 5\]

Example 5
Solve the equation,

\[ \frac{p}{2} + 3 = 7 \]

**Solution**

\textit{EITHER}  

[Subtract 3 from both sides] \[\frac{p}{2} = 4\]

[Multiply both sides by 2] \[p + 6 = 14\]

\textit{OR}  

[Multiply both sides by 2] \[p = 8\]

[Subtract 6 from both sides] \[p = 8\]

Example 6
Solve the equation

\[ \frac{p + 3}{2} = 7 \]

**Solution**

[Multiply both sides by 2] \[p + 3 = 14\]

[Subtract 3 from both sides] \[p = 11\]
Exercises

1. Solve the following equations:
   (a) \( 3x + 2 = 17 \)  (b) \( 5x - 6 = 9 \)  (c) \( 6x - 4 = 8 \)
   (d) \( 3(x + 4) = 30 \)  (e) \( 5(2x - 3) = 15 \)  (f) \( 7 - 2x = 3 \)
   (g) \( 6x - 4 = 32 \)  (h) \( 6x + 7 = 1 \)  (i) \( 7x + 6 = 34 \)
   (j) \( 6x - 7 = 11 \)  (k) \( 2x + 15 = 16 \)  (l) \( 8 - 2x = 5 \)
   (m) \( 38 = 3y + 2 \)  (n) \( 35 = 5(7 + 2p) \)  (o) \( 56 = 7(2 - 3q) \)

2. Solve the following equations:
   (a) \( \frac{x}{2} + 5 = 9 \)  (b) \( 14 = \frac{x}{3} - 8 \)  (c) \( \frac{y}{5} - 9 = -2 \)
   (d) \( \frac{z}{4} + 8 = 3 \)  (e) \( 7 = \frac{p}{4} - 6 \)  (f) \( \frac{x + 5}{2} = 9 \)
   (g) \( 14 = \frac{x - 8}{3} \)  (h) \( \frac{y - 9}{5} = -2 \)  (i) \( \frac{z + 8}{4} = 3 \)
   (j) \( 7 = \frac{p - 6}{4} \)  (k) \( \frac{2x}{3} + 1 = 9 \)  (l) \( \frac{5x}{4} - 7 = 3 \)

3. Solve the following equations:
   (a) \( 2x + 3 = x + 10 \)  (b) \( 6x - 2 = 4x + 7 \)
   (c) \( 16x - 7 = 8x + 17 \)  (d) \( 11x + 2 = 8x + 7 \)
   (e) \( x + 1 = 2(x - 1) \)  (f) \( 3(x + 4) = 5(x - 2) \)
   (g) \( 9(x + 7) = 2(5x - 7) \)  (h) \( 3(2x - 1) = 4(3x - 4) \)

4. The formula \( F = 32 + \frac{9C}{5} \) can be used to convert temperatures from degrees Celsius (\( C \)) to degrees Fahrenheit (\( F \)).
   (a) Copy and complete the following solution to calculate the value of \( C \) when \( F \) is 86 °F:
\[ F = 32 + \frac{9C}{5} \]

[Substitute 86 for \( F \)]

\[ 86 = 32 + \frac{9C}{5} \]

[Subtract 32 from both sides]

\[ = \]

[Multiply both sides by 5]

\[ = \]

[Divide both sides by 9]

\[ C = \]

(b) Calculate the value of \( C \) when \( F \) is 41, using the process as in part (a).

(c) Calculate the value of \( C \) when \( F \) is 23.

5. The formula \( p = 2(x + y) \) can be used to work out the perimeter of a rectangle with sides \( x \) and \( y \). Use the same approach as in question 4, to set up and solve an equation to calculate the value of \( x \), if \( p = 50 \) and \( y = 8 \).

6. A formula states that \( v = u + at \). Set up and solve an equation to determine the value of \( a \), if,

(a) \( v = 10, \ u = 3 \) and \( t = 5 \),

(b) \( v = 2, \ u = 5 \) and \( t = 3 \).

7. The perimeter of the rectangle shown is 16 cm. Calculate the value of \( x \).

8. The perimeter of the triangle shown is 23 cm. Calculate the value of \( x \).
9. The area of the rectangle shown is $19.5 \text{ cm}^2$. Determine the value of $x$.

10. The following two rectangles have the same areas:

(a) Determine the value of $x$.

(b) Write down the lengths of the two rectangles.

### 12.5 Non-Linear Equations

Solutions to non-linear equations are not always possible with the methods we have been using for linear equations: sometimes it is necessary to use a method called **trial and improvement**, where you make a sensible first guess at the solution, and then you try to improve your estimate. The method is illustrated in the examples which follow.

#### Example 1

Solve the equation,

$$x^3 = 100$$

giving your answer correct to 1 decimal place.

#### Solution

If we substitute $x = 4$ into the expression $x^3$, we get $4^3 = 64$, which is *less* than 100.

If we substitute $x = 5$, we get $5^3 = 125$, which is *more* than 100.

This tells us that there is a solution between 4 and 5.

We now improve our solution. A good way to record the values that you calculate is to use a table, and to make comments as you go:
From the table we can see that 4.6 is too low, and 4.65 is too high, so the solution is between 4.6 and 4.7.

So, we may write

\[ 4.6 < x < 4.65 \]

The statement means that, when it is rounded, \( x \) will be 4.6 correct to one decimal place:

\[ x = 4.6 \text{ (to 1 d.p.)} \]

Note If we need the solution to a greater degree of accuracy, then we continue the process and extend the table.

### Example 2

Use trial and improvement to solve the equation \( x^3 + x = 20 \), giving your answer correct to 2 decimal places.

### Solution

<table>
<thead>
<tr>
<th>( x )</th>
<th>( x^3 + x )</th>
<th>Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>10</td>
<td>2 is too low</td>
</tr>
<tr>
<td>3</td>
<td>30</td>
<td>3 is too high: solution is between 2 and 3</td>
</tr>
<tr>
<td>2.5</td>
<td>18.125</td>
<td>2.5 is too low</td>
</tr>
<tr>
<td>2.6</td>
<td>20.176</td>
<td>2.6 is too high: solution is between 2.5 and 2.6</td>
</tr>
<tr>
<td>2.55</td>
<td>19.131</td>
<td>2.55 is too low</td>
</tr>
<tr>
<td>2.58</td>
<td>19.754</td>
<td>2.58 is too low</td>
</tr>
<tr>
<td>2.59</td>
<td>19.964</td>
<td>2.59 is too low: solution is between 2.59 and 2.6</td>
</tr>
<tr>
<td>2.595</td>
<td>20.070</td>
<td>2.595 is too high</td>
</tr>
</tbody>
</table>
From the table we can see that the solution of $x$ is between 2.59 and 2.595, so

$$2.59 < x < 2.595$$

When we round off, $x = 2.59$ correct to 2 decimal places.

### Exercises

1. Use the trial and improvement method to solve the following equations for positive $x$:

   (a) $x^2 = 15$ to 1 decimal place,

   (b) $x^2 + x = 28$ to 1 decimal place,

   (c) $x^4 + 5 = 80$ to 1 decimal place,

   (d) $\frac{6}{x^2} = 0.1$ to 1 decimal place.

2. (a) Show that the equation $x^3 - x^2 = 2$ has a solution between 1 and 2.

   (b) Use trial and improvement to solve the equation $x^3 - x^2 = 2$, giving your solution correct to 2 decimal places.

3. (a) Show that the equation $x^2 + 2x + 3 = 15$ has a solution between 2 and 3.

   (b) Use trial and improvement to solve the equation $x^2 + 2x + 3 = 15$, giving your answer correct to 2 decimal places.

4. Use a trial and improvement method to solve $x^3 - 5x - 1 = 0$ for positive $x$, giving your answer correct to 2 decimal places.

### 12.6 Changing the Subject of a Formula

We say that $v$ is the subject of the formula $v = u + at$.

The formula can be rearranged so that $a = \frac{v - u}{t}$, and we now say that $a$ is the subject of the formula. When rearranging a formula you must use the same approach as when you solve equations.

**Note** When giving your solution at the end, remember to write the subject of the formula on the left hand side of the equation.
Example 1
Make $x$ the subject of each of the following formulae:

(a) $y = x + 8$  
(b) $y = 2x - 4$

Solution

(a) $y = x + 8$

[Subtract 8 from both sides] $y - 8 = x$  
$x = y - 8$

(b) $y = 2x - 4$

[Add 4 to both sides] $y + 4 = 2x$

[Divide both sides by 2] $\frac{y + 4}{2} = x$  
$x = \frac{y + 4}{2}$

Example 2
Make $t$ the subject of each of the following equations:

(a) $v = u + at$  
(b) $p = k(b + t)$

Solution

(a) $v = u + at$

[Subtract $u$ from both sides] $v - u = at$

[Divide both sides by $a$] $\frac{v - u}{a} = t$  
$t = \frac{v - u}{a}$

(b) $p = k(b + t)$

EITHER

[Multiply out the brackets]  
$p = kb + kt$

[Subtract $kb$ from both sides] $p - kb = kt$

[Divide both sides by $k$] $\frac{p - kb}{k} = t$  
$t = \frac{p - kb}{k}$

OR

[Divide both sides by $k$]

$p \div k = b + t$

[Subtract $b$ from both sides]  
$\frac{p}{k} - b = t$  
$t = \frac{p}{k} - b$
Note that these two formulae are equivalent, even though they look different, because

\[
\frac{p - kb}{k} = \frac{p}{k} - \frac{kb}{k} = \frac{p}{k} - b
\]

Exercises

1. Make \( x \) the subject of each of the following formulae:
   (a) \( y = x - 2 \)  
   (b) \( y = x + 7 \)  
   (c) \( y = 4x \)
   (d) \( y = \frac{x}{3} \)  
   (e) \( y = 2x + 1 \)  
   (f) \( y = 4x - 3 \)
   (g) \( y = 2(x + 3) \)  
   (h) \( y = 3(x - 4) \)  
   (i) \( y = mx \)
   (j) \( y = x + a \)  
   (k) \( y = kx - c \)  
   (l) \( y = ax + b \)

2. (a) Make \( a \) the subject of the formula \( y = ax + b \).
   (b) Make \( b \) the subject of the formula \( y = ax + b \).

3. If \( y = \frac{3x - 7}{2} \), express \( x \) in terms of \( y \).

4. The formula \( F = 32 + \frac{9C}{5} \) is used for converting temperatures for degrees Celsius (\( C \)) to degrees Fahrenheit (\( F \)). Make \( C \) the subject of this formula.

5. The perimeter of the triangle shown is \( p = a + b + c \)
   (a) Make \( a \) the subject of this formula.
   (b) Make \( c \) the subject of this formula.

6. (a) Complete the following formula for the perimeter of the rectangle shown:
   \( p = 2w + \ldots \)
   (b) Make \( w \) the subject of your formula.
(c) Complete the following formula for the area of the rectangle:

\[ A = \ldots \ldots \ldots \] 

(d) Make \( l \) the subject of your formula.

7. (a) Write down a formula for the perimeter, \( p \), of the shape shown.

(b) Make \( x \) the subject of your formula.

(c) Make \( l \) the subject of your formula

(d) Make \( b \) the subject of your formula.

8. The area of the trapezium shown is given by the formula

\[ A = \frac{h}{2}(a + b) \]

(a) Make \( h \) the subject of the formula.

(b) Make \( a \) the subject of the formula.

9. Write down a formula for the perimeter of the shape shown, and then make \( x \) the subject of the formula.

10. (a) Write down a formula for the shaded area, \( A \), in the diagram shown.

(b) Make \( x \) the subject of the formula.

(c) Make \( y \) the subject of the formula.
13 Money and Time

13.1 Money

In this section we revise basic arithmetic, working with money.

Example 1
(a) What is the cost of 7 packets of crisps costing 24p each?
(b) How much change do you get from £5 when paying for these crisps?

Solution
(a) The cost of the crisps is found by multiplying 24 by 7.

\[
\begin{array}{c}
24 \\
\times \ 7 \\
\hline
168
\end{array}
\]

The cost is 168p or £1.68.

(b) The change is found by subtracting £1.68 from £5.00.

\[
\begin{array}{c}
5.00 \\
- \ 1.68 \\
\hline
3.32
\end{array}
\]

The change is £3.32.

Example 2
Joshua buys a cheeseburger costing £1.59, a portion of chips costing 99p and a drink costing £1.15.
(a) How much does he spend?
(b) How much change does he get from a £10 note?

Solution
We add the three amounts, remembering to change the cost of the chips from pence into pounds.

\[
\begin{array}{c}
1.59 \\
1.15 \\
+ \ 0.99 \\
\hline
3.73
\end{array}
\]

He spends a total of £3.73.
Example 3
5 boys are paid £38.60 for clearing rubbish from a garden. They share the money equally. How much does each boy receive?

Solution
We divide the total amount earned by the number of boys.

\[
\frac{38.60}{5} = 7.72
\]

Each boy receives £7.72.

Example 4
Mandy buys 8 'Candichoc' bars. They cost a total of £3.04. How much does each 'Candichoc' bar cost?

Solution
We divide the total cost by the number of bars.

\[
\frac{3.04}{8} = 0.38
\]

Each 'Candichoc' bar costs 38p.

Exercises
1. Anthony pays £1.35 to swim at a sports centre. He then buys a drink costing 79p and a packet of crisps costing 27p from the sports centre café.
   (a) How much does he spend altogether?
   (b) How much money does he have left if he had £6.30 when he entered the sports centre?

2. A family buys 3 children's meals that cost £1.99 each and 2 value meals that cost £3.49 each. How much does the family spend altogether?
3. Jamil wants to buy a bike that costs £249.99. He has saved £192.50. How much more does he need to save before he can buy the bike?

4. A teacher buys a chocolate bar for each child in her class. The bars cost 34p each. There are 31 children in her class.
   (a) How much does she spend?
   (b) How much change does she get from a £20 note?

5. A tutor group raises £86.28 for charity. They decide to divide the money equally between 4 charities.
   (a) What amount do they give to each charity?
   (b) How much extra would they have to raise for each of the charities to be given £28?

6. Tickets for a school play cost £1.20 for children and £2.10 for adults. What would be the total cost of tickets for:
   (a) 2 adults and 4 children,
   (b) 3 adults and 2 children?

7. If £40.92 is divided equally between 12 people, how much do they each receive?

8. Three brothers divide £20 between them so that they each have exactly the same amount of money. A small amount is left over.
   (a) What is the largest amount they can each receive?
   (b) How much money is left over?

9. Six children are given a sum of money. They divide it equally so that they each receive £8.33 and there is 2p left over.
   (a) What was the sum of money they were given?
   (b) If £5 of the money had been given to a charity, what amount of money would each of the children have received?

10. Hannah wants to buy 12 bottles of lemonade for her birthday party. At her local supermarket, lemonade is on a ‘buy one, get one at half price’ special offer. The bottles cost £1.18 each.
    (a) How much does Hannah pay for the 12 bottles of lemonade?
    (b) How much does she save because of the special offer?
13.2 Time

In this section we revise the use of the 24-hour clock and consider problems involving time and time zones.

Example 1
Convert the following times to 24-hour clock times:
(a) 7:30 a.m.
(b) 11:45 p.m.
(c) 3:52 p.m.

Solution
(a) 0730
(b) Add 12 to the hours to give 2345.
(c) Add 12 to the hours to give 1552.

Example 2
Convert the following times from 24-hour clock to 'a.m.' or 'p.m.' times:
(a) 1426
(b) 0352
(c) 1833

Solution
(a) Subtract 12 from the hours to give 2:26 p.m.
(b) 3:52 a.m.
(c) Subtract 12 from the hours to give 6:33 p.m.

Note that a colon (:) is used to separate the hours from the minutes when using the 12-hour clock, whereas 24-hour clock times are written without a colon.

Example 3
Molly leaves Huddersfield at 1322 and arrives in London at 1805. How long does her journey take?

Solution  Method A
From 1322 to 1722 is 4 hours.
From 1722 to 1805 is 43 minutes.
Her journey takes 4 hours 43 minutes.
Example 4

The time in the United Arab Emirates is 4 hours ahead of the time in the UK.

(a) What is the time in the United Arab Emirates when it is 3:00 p.m. in the UK?

(b) If it is 2:45 p.m. in the United Arab Emirates, what is the time in the UK?

Solution

(a) The time in the United Arab Emirates is 4 hours ahead, so it is 7:00 p.m.

(b) Four hours behind 2:45 p.m. is 10:45 a.m.

Exercises

1. Convert the following times to 24-hour clock times:

   (a) 6:45 a.m.  (b) 6:45 p.m.  (c) 2:20 p.m.
   (d) 11:40 p.m. (e) 10:30 a.m. (f) 10:15 p.m.

2. Write the following 24-hour clock times in 12-hour clock times, using 'a.m.' or 'p.m.::

   (a) 1642   (b) 0832   (c) 1042
   (d) 2236   (e) 2318   (f) 1520

3. Which of the 24-hour clock times below are not possible times. Explain why.

   (a) 1372   (b) 1758
   (c) 2302   (d) 2536
4. David gets on a train at 0845 and gets off at 1132. For how long is he on the train?

5. A journey starts at 1532 and ends at 1830. How long does the journey take?

6. Marco boards a ferry at 1842 and gets off at 0633 the next day. For how long is he on the ferry?

7. In Venezuela the time is 4 hours behind the time in the UK.
   (a) What is the time in Venezuela when it is 3:00 p.m. in the UK?
   (b) What is the time in the UK when it is 2:30 p.m. in Venezuela?
   (c) What is the time in the UK when it is 11:15 p.m. in Venezuela?

8. The time in Norway is 1 hour ahead of the UK. It takes 3 1/2 hours to fly from the UK to Norway.
   (a) A plane leaves the UK at 10:15 a.m. (UK time). What is the time in Norway when it lands there?
   (b) The plane flies back and lands in the UK at 7:22 p.m. (UK time). At what time did the plane leave Norway?

9. The time in Paraguay is 4 hours behind the UK. The time in Macao is 8 hours ahead of the UK.
   (a) What is the time in Macao when it is 6:00 a.m. in Paraguay?
   (b) What is the time in Paraguay when it is 3:30 p.m. in Macao?
   (c) What is the time in Macao when it is 8:30 p.m. in Paraguay?

10. A ferry takes 26 1/2 hours to travel from the UK to Spain. The time in Spain is 1 hour ahead of the UK. When do you arrive in Spain if you leave the UK at:
    (a) 0830 on Monday   (b) 1742 on Friday   (c) 2342 on Sunday?

13.3 Time and Money

In this section we consider problems that involve both time and money.

Example 1

One day, Zoe works from 0930 until 1800. She is paid £5.20 per hour. How much does she earn for her day's work?
Solution

From 0930 until 1800 is $8\frac{1}{2}$ hours, so Zoe earns $£5.20 \times 8.5$.

Now, 

\[
\begin{array}{c}
520 \\
\times 85 \\
\hline
2600 \\
41600 \\
\hline
44200
\end{array}
\]

So $5.20 \times 8.5 = 44.200$, and Zoe earns £44.20.

Example 2

Robert works 40 hours each week and is paid £5.10 per hour.

He is given a 5% pay rise.

How much more does he earn per week after his pay rise?

Solution

Each week, Robert earns

\[
40 \times £5.10 = £204.00 \\
\times 40 \\
\hline
20400
\]

His increase each week

\[
5\% \text{ of } £204 = \frac{5}{100} \times £204 \\
= £10.20
\]

Example 3

Esther is paid £4.50 per hour. She can work for up to 30 hours per week.

(a) What is the maximum amount of money she can earn in a week?

(b) How many hours should she work if she wants to earn £90?

Solution

(a) The most she can earn in one week is

\[
30 \times £4.50 = £135 \\
\times 30 \\
\hline
13500
\]
Exercises

1. The following table shows the times that people in a factory work on one day, and the rate they are paid per hour.

<table>
<thead>
<tr>
<th>Start Work</th>
<th>Finish Work</th>
<th>Hourly Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>Janice</td>
<td>0830</td>
<td>1530</td>
</tr>
<tr>
<td>Martin</td>
<td>0745</td>
<td>1415</td>
</tr>
<tr>
<td>Gail</td>
<td>0950</td>
<td>1720</td>
</tr>
</tbody>
</table>

How much does each person earn on this day?

2. Des can choose between two jobs:
   - *Job A* pays £3.80 per hour for 40 hours per week,
   - *Job B* pays £4.50 per hour for 32 hours per week.

For which job will Des earn the most money per week?

3. Heidi works as a cleaner at a hotel. She is paid £4.20 per hour. One day she starts work at 0645 and finishes at 1045. How much does she earn on that day?

4. Briony earns £5 per hour working 12 hours per week in an evening job.
   (a) How much does she earn per week?
   (b) If she is given a 6% pay rise, how much does she now earn each week?

5. Bill works the following hours in one week:
   - Monday 0745 to 1300
   - Tuesday 1400 to 2315
   - Wednesday 0630 to 1245
   - Thursday 0745 to 1430
   - Friday 1300 to 2330

He is paid £6.50 per hour.

   (a) How many hours does he work during the week?
   (b) How much does he earn for the week's work?
6. Kelly works from 0850 until 1400 on 6 days each week. She earns £4.30 per hour.
   (a) How many hours does she work per week?
   (b) How much does she earn per week?

   Kelly is given a 10% pay rise.
   (c) How much does she now earn per week?

7. In a year, Tony works 20 hours per week for 46 weeks and is paid a total of £5520.
   (a) How many hours does he work per year?
   (b) How much is he paid per hour?
   (c) If his wages are increased by 2%, how much will he now earn per year?

8. Sara works 30 hours per week, for which she is paid £135.
   (a) How much is she paid per hour?

   Her earnings increase to £140.40 per week.
   (b) How much is she now paid per hour?
   (c) Calculate the percentage increase in her earnings.

9. Ali is paid £14.70 for working from 0845 until 1215.
   (a) How much is he paid per hour?
   (b) How much would he be paid for working from 0840 until 1300?
   (c) What would be his hourly rate of pay, if it was increased by 3%? Give your answer to the nearest pence.

10. Karen is paid £3.50 per hour for the first 40 hours she works in a week. She is paid an extra 25% per hour for any additional hours she works.
    How much does she earn for the week if she works the hours listed below:

    Monday 0855 to 1650
    Tuesday 0840 to 1710
    Wednesday 0915 to 1805
    Thursday 0855 to 1905
    Friday 0900 to 1835

    Give your answer to the nearest pence.
14 Straight Line Graphs

14.1 Coordinates

You will have used coordinates in Unit 3 of Book Y7A. In this section, we revisit coordinates before starting work on lines and graphs.

Remember that the first number is the x-coordinate and the second number is the y-coordinate.

Example 1

What are the coordinates of the points marked on the following grid:

Solution

The coordinates are:

A (8, 7)
B (9, –5)
C (–10, –6)
D (–5, 9)

Example 2

The coordinates of the corners of a shape are (2, 4), (4, 1), (2, –2), (–2, –2), (–4, 1) and (–2, 4).

(a) Draw the shape.
(b) What is the name of the shape?
Solution

(a) The shape has six sides and is called a hexagon.

Exercises

1. Write down the coordinates of each of the points marked on the following axes:
2. (a) Plot the points with coordinates \((3, -2), (-1, 6)\) and \((-5, -2)\).
(b) Join the points to form a triangle.
(c) What type of triangle have you drawn?

3. (a) Plot the points with coordinates \((-1, 4), (2, 5), (5, 4)\) and \((2, -1)\).
(b) Join these points, in order, to form a shape.
(c) What is the name of the shape that you have drawn?

4. The coordinates of 3 corners of a square are \((3, 1), (-1, 1)\) and \((3, -3)\). What are the coordinates of the other corner?

5. The coordinates of 3 corners of a rectangle are \((-1, 6), (-4, 6)\) and \((-4, -5)\). What are the coordinates of the other corner?

6. A shape has corners at the points with coordinates \((3, -2), (6, 2), (-2, 2)\) and \((-5, -2)\).
(a) Draw the shape.
(b) What is the name of the shape?

7. A shape has corners at the points with coordinates \((3, 1), (1, -3), (3, -7)\) and \((5, -3)\).
(a) Draw the shape.
(b) What is the name of the shape?

8. (a) Join the points with the coordinates below, in order, to form a polygon:
\((-5, 0), (-3, 2), (-1, 2), (1, 0), (1, -2), (-1, -4), (-3, -4)\) and \((-5, -2)\).
(b) What is the name of the polygon?

9. Three of the corners of a parallelogram have coordinates \((1, 5), (4, 4)\) and \((6, -3)\).
(a) Draw the parallelogram.
(b) What are the coordinates of the other corner?

10. Ben draws a pattern by joining, in order, the points with the following coordinates:
\((-2, 1), (-2, 2), (0, 2), (0, -1), (-4, -1), (-4, 4), (2, 4)\) and \((2, -3)\).
What are the coordinates of the next three points he would use?
14.2 Plotting Points on Straight Lines

In this section we plot points that lie on a straight line, and look for relationships between the coordinates of these points.

Example 1

(a) Plot the points with coordinates:
   \((1, 2), \ (2, 3), \ (3, 4), \ (4, 5)\) and \((5, 6)\).

(b) Draw a straight line through these points.

(c) Describe how the \(x\)- and \(y\)-coordinates of these points are related.

Solution

(a) The points are plotted below:

(b) A straight line can be drawn through these points:

(c) The \(y\)-coordinate is always one more than the \(x\)-coordinate, so we can write \(y = x + 1\).
Example 2
(a) Plot the points with coordinates:
   \((0, 0)\), \((1, 3)\), \((3, 9)\) and \((5, 15)\).
(b) Draw a straight line through these points.
(c) Write down the coordinates of two other points on this line.
(d) Describe how the \(x\)- and \(y\)-coordinates are related.

Solution
(a) The points are plotted below: 
(b) A line can then be drawn through these points:

(c) The points \((2, 6)\), and \((4, 12)\) also lie on the line (and many others).
(d) The \(y\)-coordinate is 3 times the \(x\)-coordinate. So we can write \(y = 3x\)
Exercises

1. (a) Plot the points with coordinates
   \((0, 4), (1, 5), (3, 7)\) and \((5, 9)\).
   (b) Draw a straight line through the points.
   (c) Write down the coordinates of 3 other points that lie on this line.

2. (a) Plot the points with coordinates
   \((0, 6), (2, 4), (3, 3)\) and \((5, 1)\)
   and draw a straight line through them.
   (b) On the same graph as used for question 2(a), plot the points with coordinates
   \((1, 8), (2, 7), (5, 4)\) and \((7, 2)\)
   and draw a straight line through them.
   (c) Copy and complete the sentence:
   "These two lines are p.....................  ".

3. (a) Plot the points with coordinates
   \((2, 6), (3, 5), (4, 4)\) and \((7, 1)\)
   and draw a straight line through them.
   (b) On the same set of axes, plot the points with coordinates
   \((0, 1), (1, 2), (3, 4)\) and \((5, 6)\)
   and draw a straight line through them.
   (c) Copy and complete this sentence:
   "These two lines are p.....................  ".

4. (a) Plot the points with coordinates
   \((1, 1), (2, 2), (4, 4)\) and \((5, 5)\)
   and draw a straight line through them.
   (b) Write down the coordinates of two other points on the line.
   (c) Describe the relationship between the \(x\)- and \(y\)-coordinates.

5. The points \((1, 3), (2, 4), (3, 5)\) and \((5, 7)\) lie on a straight line.
   (a) Plot these points and draw the line.
   (b) Write down the coordinates of 3 other points on the line.
   (c) Describe the relationship between the \(x\)- and \(y\)-coordinates.
6. (a) Plot the points (0, 5), (2, 3), (4, 1) and (5, 0). Draw a straight line through them.
   (b) Write down the coordinates of two other points on the line.
   (c) The relationship between the x- and y-coordinates can be written as $x + y = \underline{\hspace{2cm}}$. What is the missing number?

7. (a) Plot the points with coordinates
       (–3, –4), (–1, –2), (1, 0), (4, 3)
   (b) Draw a straight line graph through these points.
   (c) Describe the relationship between the x- and y-coordinates.

8. The points with coordinates (–2, –4), (2, 4), (3, 6) and (4, 8) lie on a straight line.
   (a) Draw the line.
   (b) Describe the relationship between the x- and y-coordinates of points on the line.

9. The points with coordinates (–6, –3), (–1, 2), (2, 5) and (4, 7) lie on a straight line.
   (a) Draw the line.
   (b) Complete the missing numbers in the coordinates of other points that lie on the line:
       (–7, \underline{\hspace{2cm}}), (\underline{\hspace{2cm}}, –1), (3, \underline{\hspace{2cm}}), (\underline{\hspace{2cm}}, 4), (100, \underline{\hspace{2cm}})
   (c) Describe the relationship between the x- and y-coordinates of the points on the line.
   (d) Will the point with coordinates (25, 27) lie on the line? Give a reason for your answer.

10. Each set of points listed below lies on a straight line. Plot the points, draw the line, and complete the statement about the relationship between the x- and y-coordinates.
    (a) (1, 6), (3, 4), (8, –1) \hspace{1cm} x + y = \underline{\hspace{2cm}}
    (b) (–4, 2), (–1, 5), (3, 9) \hspace{1cm} y = x + \underline{\hspace{2cm}}
    (c) (–2, –8), (0, 0), (3, 12) \hspace{1cm} y = \underline{\hspace{2cm}} x
    (d) (–4, –6), (–1, –3), (3, 1) \hspace{1cm} y = x – \underline{\hspace{2cm}}
14.3 Plotting Graphs Given Their Equations

In this section we see how to plot a graph, given its equation. We also look at how steep it is and use the word gradient to describe this. There is a simple connection between the equation of a line and its gradient, which you will notice as you work through this section.

**Gradient of a Line**

\[
\text{Gradient} = \frac{\text{Rise}}{\text{Step}}
\]

You can draw any triangle using the sides to determine the rise and step, but the triangle must have one side horizontal and one side vertical.

**Example 1**

Determine the gradient of each of the following lines:

(a) \(\) \(\) \(\) \(\) \(\)

**Solution**

(a) \(\)

\[
\text{Gradient} = \frac{\text{Rise}}{\text{Step}}
\]

\[
= \frac{2}{1}
\]

\[
= 2
\]
14.3

Example 2

(a) Complete the table below for \( y = 2x + 1 \).

<table>
<thead>
<tr>
<th>( x )</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(b) Use the information in the table to plot the graph with equation \( y = 2x + 1 \).

Solution

(a) Complete the table below for \( y = 2x + 1 \).

<table>
<thead>
<tr>
<th>( x )</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>-3</td>
<td>-1</td>
<td>1</td>
<td>3</td>
<td>5</td>
</tr>
</tbody>
</table>
(b) The points

\((-2, -3), (-1, -1), (0, 1)\)

\((1, 3)\) and \((2, 5)\)

can then be plotted, and a straight line drawn through these points.

Example 3

(a) Draw the graph of the line with equation \(y = x + 1\).
(b) What is the gradient of the line?

Solution

(a) The table shows how to calculate the coordinates of some points on the line.

<table>
<thead>
<tr>
<th>(x)</th>
<th>(-3)</th>
<th>(-2)</th>
<th>(-1)</th>
<th>(0)</th>
<th>(1)</th>
<th>(2)</th>
<th>(3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(y)</td>
<td>(-2)</td>
<td>(-1)</td>
<td>(0)</td>
<td>(1)</td>
<td>(2)</td>
<td>(3)</td>
<td>(4)</td>
</tr>
</tbody>
</table>

The points with coordinates \((-3, -2), (-2, -1), (-1, 0), (0, 1), (1, 2), (2, 3)\) and \((3, 4)\) can then be plotted and a line drawn as shown:

(b) To calculate the gradient of the line, draw a triangle under the line as shown in the diagram on the next page. The triangle can be of any size, but must have one horizontal side and one vertical side.
Exercises

1. Which of the following lines have a *positive* gradient and which have a *negative* gradient:

![Diagram of lines A, B, C, D, E, F with coordinates and gradients](image-url)
2. Determine the gradient of each of the following lines:

(a) 
(b) 
(c) 
(d) 
(e) 
(f) 
(g) 
(h) 

3. Determine the gradient of each of the following lines:

(a) 
(b) 
(c) 
(d) 

4. (a) Copy and complete the following table for \( y = x - 2 \).

<table>
<thead>
<tr>
<th>( x )</th>
<th>-3</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td></td>
<td></td>
<td></td>
<td>-2</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(b) Draw the line with equation \( y = x - 2 \).

5. (a) Copy and complete the following table for \( y = 2x + 3 \).

<table>
<thead>
<tr>
<th>( x )</th>
<th>-3</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>9</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(b) Draw the line with equation \( y = 2x + 3 \).
6. (a) Draw the line with equation $y = 2x - 1$.
   (b) Determine the gradient of this line.

7. (a) Draw the line with equation $y = \frac{1}{2}x + 2$.
   (b) Determine the gradient of this line.

8. (a) Draw the lines $y = 3x + 1$ and $y = 4x - 5$.
   (b) Determine the gradient of each of these lines.

9. Without drawing the lines, state the gradients of the lines with the following equations:
   (a) $y = 2x + 4$
   (b) $y = 3x - 9$
   (c) $y = 10x + 1$
   (d) $y = 5x + 3$

10. (a) Draw the lines and equations $y = 2x + 1$ and $y = 3x - 2$.
    (b) Write down the coordinates of the point where these two lines cross.

11. Determine the coordinates of the point where the lines $y = x + 3$ and $y = 7 - x$ cross.

12. (a) Draw the line with equation $y = 6 - 2x$.
    (b) Explain why the gradient of this line is $-2$.

13. (a) Explain why the lines with equations $y = 2 - 2x$ and $y = 5 - 2x$ are parallel.
    (b) Write down the equation of another line that would be parallel to these lines.
    (c) Draw all three lines.
14.4 The Equation of a Straight Line

In this section we examine how the equation of a straight line contains information about the gradient of the line and the point where it crosses the \( y \)-axis.

The intercept is \( c \), that is the point where the line crosses the \( y \)-axis.

The gradient is \( m \), where

\[
m = \frac{\text{Rise}}{\text{Step}}
\]

The equation of a straight line is \( y = mx + c \).

Example 1

(a) Determine the equation of the line shown below:

Solution

First note that the intercept is 2, so we write \( c = 2 \).

Next calculate the gradient of the line.

Note that the rise is \(-6\), as the line is going down as you move from left to right.
The equation of a straight line is \( y = mx + c \), so here, with \( m = -1 \) and \( c = 2 \), we have

\[
y = -x + 2
\]

or

\[
y = 2 - x.
\]

**Reminder**

Recall that \( -1 \times x = -1 \times x \) is written as \( -x \) for speed and convenience.

**Exercises**

1. (a) Draw the line with equation \( y = 2x + 3 \).
   (b) Determine the gradient of this line.
   (c) What is the intercept of this line?

2. (a) Draw the lines with equations \( y = x \), \( y = -x \), \( y = 2x \) and \( y = -3x \).
   (b) Determine the gradient of each of these lines.
   (c) What is the intercept of each of these lines?

3. The points with coordinates \((-2, 3), \ (0, 5) \) and \((3, 8)\) lie on a straight line.
   (a) Plot the points and draw the line.
   (b) Determine the gradient of the line.
   (c) What is the intercept of the line?
   (d) Write down the equation of the line.
4. Determine the equation of each of the lines shown below:

(a) 
(b) 
(c) 
(d) 

5. Copy and complete the following table:

<table>
<thead>
<tr>
<th>Equation</th>
<th>Gradient</th>
<th>Intercept</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y = 2x + 7 )</td>
<td>8</td>
<td>-2</td>
</tr>
<tr>
<td>( y = 8 - 3x )</td>
<td>7</td>
<td>-5</td>
</tr>
<tr>
<td></td>
<td>-3</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>-5</td>
<td>-2</td>
</tr>
</tbody>
</table>
6. (a) Draw the lines with equations \( y = x + 1 \), \( y = 1 - x \), \( y = 2x + 1 \) and \( y = 3x + 1 \) on the same set of axes.
(b) Explain why these lines all pass through the same point on the \( y \)-axis.

7. The points with coordinates \((-2, -6)\), \((0, 0)\) and \((3, 9)\) all lie on a straight line.
(a) What is the gradient of the line?
(b) What is the intercept of the line?
(c) What is the equation of the line?

8. Draw lines which have:
(a) gradient 2 and intercept 3,
(b) gradient \( \frac{1}{2} \) and intercept 1,
(c) gradient \(-4\) and intercept 7.

14.5 The Equation of a Line Given Two Points

If you know the coordinates of two points on a line, it is possible to determine its equation without drawing the line.

If a line passes through the points with coordinates \((x_1, y_1)\) and \((x_2, y_2)\), the gradient, \(m\), of the line is given by
\[
m = \frac{y_2 - y_1}{x_2 - x_1}
\]

**Example 1**

Determine the equation of the line that joins the points with coordinates \((4, 8)\) and \((10, 11)\).

**Solution**

First determine the gradient of the line:
\[
m = \frac{11 - 8}{10 - 4} = \frac{3}{6} = \frac{1}{2}
\]
Now the equation must be \( y = \frac{1}{2} x + c \).

To determine \( c \), use the values of \( x \) and \( y \) from one of the points. Here \( x = 4 \) and \( y = 8 \), and substitute in the equation, giving:

\[
8 = \frac{1}{2} \times 4 + c \\
8 = 2 + c \\
c = 6
\]

So the equation of the line is given by \( y = \frac{1}{2} x + 6 \).

**Exercises**

1. A straight line joins the points with coordinates \((1, 1)\) and \((4, 7)\).
   (a) Determine the gradient of the line.
   (b) Determine the equation of the line.

2. Determine the equation of the line that passes through the points \((0, 0)\) and \((3, 21)\).

3. Explain why a line that passes through the point \((0, 0)\) and any other point has equation \( y = mx \).

4. Determine the equation of a straight line that passes through the following pairs of points:
   (a) \((0, 1)\) and \((5, 16)\)  
   (b) \((3, 20)\) and \((7, 32)\)  
   (c) \((0, 100)\) and \((50, 0)\)  
   (d) \((-1, 9)\) and \((3, -3)\)  
   (e) \((-6, -4)\) and \((10, 28)\)  
   (f) \((-6, -2)\) and \((-2, -9)\)

5. A line has gradient \(-4\) and passes through the point with coordinates \((5, 7)\). What is the equation of the line?

6. A triangle has corners at the points with coordinates \((1, 2), (-2, 3)\) and \((0, -1)\). Determine the equations of the lines that form the sides of the triangle.

7. A parallelogram has corners at the points with coordinates \((-1, 1), (0, 3), (2, -1)\) and \((1, -3)\). Determine the equations of the lines that form the sides of the parallelogram.
15 Polygons

15.1 Angle Facts

In this section we revise some basic work with angles, and begin by using the three rules listed below:

- The angles at a point add up to $360^\circ$, e.g.
  \[
  a + b + c = 360^\circ
  \]

- The angles on a straight line add up to $180^\circ$, e.g.
  \[
  e + f = 180^\circ
  \]

- The angles in a triangle add up to $180^\circ$, e.g.
  \[
  w + x + y = 180^\circ
  \]

Example 1

Determine the size of angle $a$ in the diagram shown.

**Solution**

\[
81^\circ + 92^\circ + 100^\circ + a = 360^\circ \quad (angle \ sum \ at \ a \ point)
\]

\[
a + 273^\circ = 360^\circ
\]

\[
a = 87^\circ
\]

Example 2

Determine the size of angle $d$ in the diagram shown.

**Solution**

\[
105^\circ + 42^\circ + d = 180^\circ \quad (angle \ sum \ in \ a \ triangle)
\]

\[
147^\circ + d = 180^\circ
\]

\[
d = 33^\circ
\]
Example 3
Determine the size of angle $n$ in the diagram shown.

Solution
$$n + 27^\circ = 180^\circ \text{ (angle sum on a line)}$$
$$n = 153^\circ$$

Exercises
1. Calculate the sizes of the angles marked by letters in the following diagrams:
   (a) ![Diagram](image1)
   (b) ![Diagram](image2)
   (c) ![Diagram](image3)
   (d) ![Diagram](image4)

2. Calculate the sizes of the unknown angles in the following triangles:
   (a) ![Diagram](image5)
   (b) ![Diagram](image6)
   (c) ![Diagram](image7)
   (d) ![Diagram](image8)
3. Calculate the sizes of the angles marked by the letter $x$ in the following diagrams:

(a) 
![Diagram (a)](image)

(b) 
![Diagram (b)](image)

(c) 
![Diagram (c)](image)

(d) 
![Diagram (d)](image)

4. The diagram shows an isosceles triangle.
What are the sizes of the two angles marked $a$ and $b$?

![Diagram (4)](image)

5. Calculate the sizes of the angles marked $a$ and $b$ in the diagram.

![Diagram (5)](image)

6. The diagram opposite shows two intersecting straight lines. Calculate the sizes of the angles marked $a$, $b$, and $c$ in the diagram.
What do you notice about angles $a$ and $c$?

![Diagram (6)](image)

7. The diagram opposite shows a rectangle and its diagonals. Calculate the sizes of the angles marked $a$, $b$, and $c$.

![Diagram (7)](image)
8. Determine the sizes of the angles marked $a$, $b$ and $c$ in the diagram shown.

9. PQR is a straight line. Determine the sizes of the angles marked $a$, $b$ and $c$ in the triangles shown.

10. Calculate the sizes of the angles marked $a$, $b$, $c$, $d$ and $e$ in the triangles shown.
15.2 Angle Properties of Polygons

In this section we calculate the size of the interior and exterior angles for different regular polygons.

The following diagram shows a regular hexagon:

In a regular polygon the sides are all the same length and the interior angles are all the same size.

Note that, for any polygon:

\[
\text{interior angle} + \text{exterior angle} = 180^\circ.
\]

Since the interior angles of a regular polygon are all the same size, it follows that the exterior angles are also equal to one another.

One complete turn of the hexagon above will rotate any one exterior angle to each of the others in turn, which illustrates the following result:

The exterior angles of any polygon add up to \(360^\circ\).

Example 1

Calculate the sizes of the interior and the exterior angles of a regular hexagon. Hence determine the sum of the interior angles.

Solution

The exterior angles of a regular hexagon are all equal, as shown in the previous diagram.
Therefore the exterior angle of a regular hexagon \( = \frac{360^\circ}{6} \)
\[= 60^\circ\]

So the interior angle of a regular hexagon \( = 180^\circ - 60^\circ \)
\[= 120^\circ\]

The sum of the interior angles \( = 6 \times 120^\circ \)
\[= 720^\circ\]

Example 2

The *exterior* angle of a regular polygon is \( 40^\circ \).

Calculate:

(a) the size of the *interior* angle,
(b) the number of sides of the polygon.

**Solution**

(a) Interior angle + exterior angle \( = 180^\circ \)

Interior angle \( = 180^\circ - 40^\circ \)
\[= 140^\circ\]

(b) The number of sides can be determined by dividing \( 360^\circ \) by the size of the exterior angles, giving

\[
\frac{360^\circ}{40^\circ} = 9
\]

so the polygon has \( 9 \) sides.

In a regular polygon:

\[
\begin{align*}
\text{exterior angle} & = \frac{360^\circ}{\text{the number of sides}} \\
\text{number of sides} & = \frac{360^\circ}{\text{exterior angle}}
\end{align*}
\]
Exercises

1. Calculate the size of the exterior angles of a regular polygon which has interior angles of:
   (a) 150 °
   (b) 175 °
   (c) 162 °
   (d) 174 °

2. Calculate the sizes of the exterior and interior angles of:
   (a) a regular octagon,
   (b) a regular decagon.

3. (a) Calculate the size of the interior angles of a regular 12-sided polygon.
   (b) What is the sum of the interior angles of a regular 12-sided polygon?

4. (a) What is the size of the interior angle of a regular 20-sided polygon?
   (b) What is the sum of the interior angles of a regular 20-sided polygon?

5. Calculate the size of the exterior angle of a regular pentagon.

6. The size of the exterior angle of a regular polygon is 12 °. How many sides does this polygon have?

7. Calculate the number of sides of a regular polygon with interior angles of:
   (a) (i) 150 ° (ii) 175 °
       (iii) 162 ° (iv) 174 °
   (b) Show why it is impossible for a regular polygon to have an interior angle of 123 °.

8. (a) Complete the following table for regular polygons. Note that many of the missing values can be found in the examples and earlier exercises for this unit.
(b) Describe an alternative way to calculate the sum of the interior angles of a regular polygon.

(c) Draw and measure the angles in some irregular polygons. Which of the results in the table are also true for irregular polygons?

9. The exterior angle of a regular polygon is $4^\circ$.

(a) How many sides does the polygon have?

(b) What is the sum of the interior angles of the polygon?

10. A regular polygon has $n$ sides.

(a) Explain why the exterior angles of the polygon are of size $\frac{360^\circ}{n}$.

(b) Explain why the interior angles of the polygon are $180^\circ - \frac{360^\circ}{n}$.

(c) Write an expression for the sum of the interior angles.
15.3 Symmetry

In this section we revise the symmetry of objects and examine the symmetry of regular polygons.

Example 1

Draw the lines of symmetry of each shape below:

(a) (b)

Solution

(a) The shape has 2 lines of symmetry, one horizontal and the other vertical, as shown below:

(b) The shape has 2 diagonal lines of symmetry, as shown below:

Reminder

The order of rotational symmetry is the number of times in one rotation of 360° that a shape is identical to that of its starting position.

Example 2

What is the order of rotational symmetry of each of the following shapes:

(a) (b) (c)
Example 3

A heptagon is a shape which has 7 sides.
(a) Draw a diagram to show the lines of symmetry of a regular heptagon.
(b) What is the order of rotational symmetry of a regular heptagon?

Solution

(a) A regular heptagon has 7 lines of symmetry, as shown in the following diagram:
(b) A regular heptagon has rotational symmetry of order 7.

The order of rotational symmetry and the number of lines of symmetry of any regular polygon is equal to the number of sides.

Exercises

1. Copy each of the following shapes and draw in all the lines of symmetry. For each one, state the order of rotational symmetry and mark on your copy its centre of rotation.

(a)  
(b)  
(c)  
(d)  
(e)  
(f)  

15.3
2. State the order of rotational symmetry and the number of lines of symmetry, for each of the following shapes:

(a) 
(b) 
(c) 
(d) 
(e) 
(f) 

3. Describe fully the symmetries of the following shapes:

(a) 
(b) 

4. Describe the symmetry properties of each of the following triangles:

Equilateral
Isosceles
Scalene
5. (a) How many lines of symmetry does a square have? Draw a diagram to show this information.
(b) What is the order of rotational symmetry of a square?

6. (a) Copy and complete the following table:

<table>
<thead>
<tr>
<th>Shape</th>
<th>Order of Rotational Symmetry</th>
<th>Number of Lines of Symmetry</th>
</tr>
</thead>
<tbody>
<tr>
<td>Equilateral triangle</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Square</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Regular pentagon</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Regular hexagon</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Regular heptagon (7 sides)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Regular octagon</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Regular nonagon (9 sides)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Regular decagon (10 sides)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Regular dodecagon (12 sides)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(b) What do you conclude from the information in the table?

7. Draw a shape that has no lines of symmetry, but has rotational symmetry of order 3.

8. Draw a shape that has at least one line of symmetry and no rotational symmetry.

9. Draw two regular polygons, one with an even number of sides and one with an odd number of sides. By drawing lines of symmetry on each diagram, show how the positions of the lines of symmetry differ between odd- and even-sided regular polygons.

10. Draw an irregular polygon that has both line and rotational symmetry. Show the lines of symmetry and the centre of rotation, and state its order of rotational symmetry.
## Quadrilaterals

There are many special types of quadrilaterals; the following table lists some of them and their properties.

<table>
<thead>
<tr>
<th>Quadrilateral</th>
<th>Properties</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rectangle</td>
<td>4 right angles and opposite sides equal</td>
</tr>
<tr>
<td>Square</td>
<td>4 right angles and 4 equal sides</td>
</tr>
<tr>
<td>Parallelogram</td>
<td>Two pairs of parallel sides and opposite sides equal</td>
</tr>
<tr>
<td>Rhombus</td>
<td>Parallelogram with 4 equal sides</td>
</tr>
<tr>
<td>Trapezium</td>
<td>Two sides are parallel</td>
</tr>
<tr>
<td>Kite</td>
<td>Two pairs of adjacent sides of the same length</td>
</tr>
</tbody>
</table>

### Example 1
List the quadrilaterals that have four sides all of the *same length*.

**Solution**

*Square* and *rhombus*.

### Example 2
List the quadrilaterals that do *not* have two pairs of parallel sides.

**Solution**

*Kite* and *trapezium*.

### Example 3
Which quadrilaterals have diagonals that are *perpendicular* to one another?

**Solution**

The *square*, *rhombus* and *kite* have diagonals that cross at right angles.
Exercises

1. Which quadrilaterals have diagonals that are the same length?

2. (a) Which quadrilaterals have *exactly two* lines of symmetry?
   (b) Draw diagrams to show these lines of symmetry.

3. Which quadrilaterals have rotational symmetry of order 2?

4. (a) Which quadrilaterals can have *exactly one* line of symmetry?
   (b) Draw diagrams to show them and the line of symmetry.

5. Name each of the following quadrilaterals:
   (a) ![Diagram](image1.png)
   (b) ![Diagram](image2.png)
   (c) ![Diagram](image3.png)
   (d) ![Diagram](image4.png)
   (e) ![Diagram](image5.png)
   (f) ![Diagram](image6.png)
6. Which quadrilaterals have diagonals that are *not* equal in length?

7. A quadrilateral has four sides of the same length. Copy and complete the following sentences:
   (a) The quadrilateral must be a ................. .
   (b) The quadrilateral could be a ................. if ................. .

8. (a) Which quadrilaterals have *more than one* line of symmetry?
   (b) Draw diagrams to show them and their lines of symmetry.
   (c) Which quadrilaterals have rotational symmetry of order *greater than 1*?
       List these quadrilaterals and state the order of their rotational symmetry.

9. The following flow chart is used to identify quadrilaterals:

   Which type of quadrilateral arrives at each of the outputs, A to G?
10. The following flow chart can be used to classify quadrilaterals, but some question boxes are empty. Copy and complete the flow chart.
16 Circles and Cylinders

16.1 Introduction to Circles

In this section we consider the circle, looking at drawing circles and at the lines that split circles into different parts.

A chord joins any two points on the circumference of a circle.

A diameter is a chord that passes through the centre of the circle.

A radius joins the centre of the circle to any point on the circumference of the circle.

A chord splits a circle into two segments.

The larger one is called a major segment: the smaller one is the minor segment.

The part of a circle between two radii is called a sector.

The part of the circle that forms the curved side of the sector is called an arc.
Example 1
(a) Draw a circle of radius 5 cm.
(b) Draw a radius of the circle.
(c) Draw a chord that is perpendicular to the radius and is 3.3 cm from the centre of the circle.
(d) Measure the length of the chord.

Solution
(a) First draw the circle.
(b) Then draw in a radius.
(c) Measure 3.3 cm along the radius from the centre of the circle. Draw a chord at right angles to the radius and through this point.
(d) Measure the chord, which gives 7.5 cm.

Example 2
(a) Draw a circle of radius 3 cm.
(b) Draw a chord of length 5 cm, inside the circle.
(c) Draw the perpendicular bisector of the chord.
(d) What is the length of the new chord that is formed by the perpendicular bisector?

Solution
(a) First draw the circle.
(b) Put your compass point at A, with your compass set at 5 cm. Draw an arc to find the point B. Then join the points A and B.
(c) To draw a perpendicular bisector, place your compass point at A and draw two arcs. Repeat with your compass point at B, drawing arcs of the same radius as before, so that they intersect. Draw a line through the two intersections.

(d) Measure the length of the new chord as 6 cm. The new chord is, in fact, a diameter of the circle.

Exercises

1. The diagram shows a circle with centre, O. What is the name given to each of the following lines:
   (a) O A  (b) A B  (c) B C  (d) O D
   (e) C D  (f) A C  (g) A D
2. Which of the parts of the circle shown are:
   (a) sectors,
   (b) segments,
   (c) triangles?

3. (a) Draw a circle of radius 5 cm.
   (b) Draw any chord in this circle.
   (c) Draw the perpendicular bisector of the chord.
   (d) Draw 2 other chords and their perpendicular bisectors.
   (e) Comment on the perpendicular bisectors of the chords.

4. (a) Draw a circle of radius 4 cm and a chord of length 3 cm, in the circle.
   (b) Join the ends of the chord to the centre of the circle, to form a triangle.
   (c) What length is the perimeter of the triangle?

5. The diagram shows a sketch of a triangle which is drawn inside a circle of radius 3 cm.
   (a) Draw this triangle accurately.
   (b) Determine the perimeter of the triangle.

6. The diagram shows a triangle drawn inside a circle of radius 5 cm. The line A B is a diameter.
   (a) Draw this triangle accurately.
   (b) Determine the length of A C.
   (c) Determine the size of the angle A C B.
   (d) Calculate the area of the triangle.
7. In a circle of radius 2.5 cm, draw a radius and the chord that is a perpendicular bisector to the radius. What is the length of this chord?

8. A triangle is drawn so that the 3 corners are on a circle of radius 3 cm. Two of the sides have length 5 cm.
   (a) Draw the triangle.
   (b) Determine the length of the third side.
   (c) Draw the perpendicular bisector of each side of the triangle.
   (d) How far is it along each perpendicular bisector from the side to the centre of the circle?

9. (a) Draw any triangle.
   (b) Draw the perpendicular bisector of each side.
   (c) Draw a circle with its centre at the point where the perpendicular bisectors intersect, and that passes through the three corners of the triangle.

10. Draw a circle of radius 3 cm. A chord in the circle has length 4 cm. Determine the distance from the centre of the chord to the centre of the circle.

16.2 Estimating the Circumference of a Circle

In this section we investigate the relationship between the diameter and the circumference of a circle. The circumference is the distance round the outside of a circle.
Example 1

Measure the diameter and circumference of the circle shown.

Solution

The diameter can be measured directly as 5.2 cm.

To measure the circumference, take a piece of string and lay it round the outside of the circle.

Then lay the string along a ruler and measure the circumference as 16.3 cm.

Reminder

The points shown below form a scatter graph. The line drawn on the graph is called the 'line of best fit': it is the straight line which best fits the points plotted. It does not have to go through every point; just as close to them as possible. The line should be positioned so that approximately the same number of points are above it as below it.

You will need to draw lines of best fit as you work through this unit.
Exercises

1. (a) Draw circles of radius 1 cm, 2 cm, 3 cm, 4 cm, 5 cm and 6 cm.
   (b) Measure the circumference and diameter of each circle.

2. (a) Obtain a number of circular objects, for example,
      bottle of squash,
      tin of baked beans,
      bottle of Tippex,
      roll of Sellotape, etc.
   (b) For each object, measure the diameter and the circumference.

3. (a) Draw a scatter graph to show your results from questions 1 and 2, on a set of axes like those that follow:

   ![Graph]

   (b) Explain why a line of best fit should pass through the point with coordinates (0, 0).
   (c) Draw a line of best fit.
   (d) The relationship between the circumference, \( C \), and the diameter, \( d \), is given by \( C = kd \), where \( k \) is a constant number. Use your line of best fit to determine \( k \).
16.3 Estimating the Area of a Circle

In this section we investigate how the area of a circle depends on the radius of the circle.

Example 1
Estimate the area of a circle of radius 3 cm.

Solution
The following diagram shows the circle drawn on squared paper. Complete squares are numbered 1 to 16 whilst partial squares are numbered in brackets, e.g. (17), and joined together where possible to approximately make a square, e.g. the two part squares marked (26) add to make about one whole square.

Counting the squares shows that the area is approximately 28 cm$^2$.

Exercises
1. Draw circles of radius 1 cm, 2 cm, 3 cm, 4 cm, 5 cm, 6 cm and 7 cm on squared paper. Determine the approximate area of each of the circles.

2. What is the area of a circle of radius 0 cm?

3. (a) Draw a scatter graph to show area against radius for the results that you have obtained in question 1.
   (b) Explain why it would not be sensible to draw a line of best fit through the points that you have plotted.

4. (a) Copy and complete the following table:

<table>
<thead>
<tr>
<th>Radius (cm)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>(Radius)$^2$ (cm$^2$)</td>
<td>0</td>
<td>1</td>
<td>9</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Approximate Area (cm$^2$)</td>
<td>0</td>
<td>28</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
(b) On a set of axes like those shown below, draw a scatter graph using the data from your table.

(c) Draw a line of best fit through your data points.

(d) The area, $A$, of a circle of radius $r$, can be found using the formula $A = kr^2$, where $k$ is a constant number. Use your line of best fit to determine the value of $k$.

16.4 Formulae for Circumference and Area

In the previous two sections you have found approximate formulae for both the circumference and the area of a circle. The exact formulae are given below:

$$\text{Circumference } = \pi d$$

or

$$\text{Circumference } = 2\pi r$$

$$\text{Area } = \pi r^2$$

The symbol $\pi$ (lower case Greek letter p) represents a special number called 'pi'. The value of $\pi$ has been calculated to over 1000 million decimal places; its value correct to 5 decimal places is $3.14159$.

Compare this with the gradient you obtained from your scatter graphs in the last two sections, to see how close you were.

There is a button on your calculator which you can use when doing calculations involving $\pi$, as the next examples illustrate.
Example 1
A circle has radius 6 cm. Calculate:
(a) its circumference,
(b) its area.

Solution
(a) Circumference \(= 2 \pi r\)
    \(= 2 \pi \times 6\)
    \(= 37.7 \text{ cm to 3 significant figures.}\)
(b) Area \(= \pi r^2\)
    \(= \pi \times 6^2\)
    \(= 113 \text{ cm}^2 \text{ to 3 significant figures}\)

Example 2
A circle has diameter 7 cm. Calculate:
(a) its circumference,
(b) its area.

Solution
(a) Circumference \(= \pi d\)
    \(= \pi \times 7\)
    \(= 22.0 \text{ cm to 3 significant figures.}\)
(b) Radius \(= 3.5 \text{ cm}\)
    Area \(= \pi r^2\)
    \(= \pi \times 3.5^2\)
    \(= 38.5 \text{ cm}^2 \text{ to 3 significant figures}\)

Example 3
The circumference of a circle is 18.2 cm. Calculate the length of the diameter, \(d\), of the circle.
**Solution**

\[ C = \pi d \]

18.2 = \pi \ d

\[
\frac{18.2}{\pi} = d
\]

\[ d = 5.79 \text{ cm to 3 significant figures.} \]

**Example 4**

The area of a circle is 22.8 cm\(^2\). Calculate the length of the radius, \( r \), of the circle.

**Solution**

\[ A = \pi r^2 \]

22.8 = \pi \ r^2

\[
\frac{22.8}{\pi} = r^2
\]

\[ r = \sqrt{\frac{22.8}{\pi}} \]

\[ = 2.69 \text{ cm to 3 significant figures.} \]

**Exercises**

1. A circle has radius 11 cm. Calculate:
   
   (a) its diameter,
   
   (b) its circumference,
   
   (c) its area.

2. Calculate the circumference and area of a circle with radius 8 cm.

3. Calculate the circumference and area of a circle with diameter 19 cm.
4. Copy and complete the following table:

<table>
<thead>
<tr>
<th>Radius</th>
<th>Diameter</th>
<th>Circumference</th>
<th>Area</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>24 cm</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1 cm</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6 mm</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>9 m</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>23 km</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

5. Determine the circumference and area of the circle shown:

6. A circle is cut out of a rectangular piece of card, as shown:
   (a) Calculate the area of the rectangle.
   (b) Calculate the area of the circle.
   (c) Calculate the area of the card left, when the circle has been cut out.

7. Calculate the area and perimeter of the semicircle shown:

8. The circumference of a circle is 29 cm.
   (a) Calculate the radius of the circle.
   (b) Calculate the area of the circle.
9. The area of a circle is 48 cm\(^2\). Calculate the radius and circumference of the circle.

10. Copy and complete the following table:

<table>
<thead>
<tr>
<th>Radius</th>
<th>Diameter</th>
<th>Circumference</th>
<th>Area</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>82 cm</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>19 m(^2)</td>
</tr>
<tr>
<td></td>
<td>33 m</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>44 mm</td>
<td>36 mm(^2)</td>
</tr>
</tbody>
</table>

11. A circle is cut up into sectors that can be placed side-by-side as shown in the following diagram:

If the angles of the sectors are very small, the shape formed almost becomes a rectangle. In this case, \(w\) is nearly equal to \(r\), the radius of the circle.

(a) Explain why \(l\) is approximately \(\pi r\).

(b) Use the fact that the shape is close to a rectangle to derive a formula for the area of the circle.
16.5 Problems in Context

In this section we apply the formulae for area and circumference to some problems.

Example 1

The diagram shows an arched window made in the shape of a semicircle on top of a rectangle. Calculate the area of the window.

**Solution**

Area of rectangle = \(80 \times 60\)

= \(4800 \text{ cm}^2\)

Radius of semicircle = 30 cm

Area of semicircle = \(\frac{1}{2} \times \pi \times 30^2\)

= 1414 \text{ cm}^2 \text{ (4 s.f.)}

Total area = 4800 + 1414

= 6214 \text{ cm}^2 \text{ (4 s.f.)}

Example 2

A circular disc of diameter 10 cm has a hole of diameter 4 cm cut in it. Calculate the area remaining of the large disc, as shaded in the diagram.

**Solution**

Radius of large disc = 5 cm

Area of large disc = \(\pi \times 5^2\)

= 78.54 \text{ cm}^2 \text{ (2 d.p.)}

Radius of hole = 2 cm

Area of hole = \(\pi \times 2^2\)

= 12.57 \text{ cm}^2 \text{ (2 d.p.)}

Shaded area = 78.54 – 12.57

= 65.97 \text{ cm}^2

= 66.0 \text{ cm}^2
Example 3

The diagram shows a square with sides of length 6 cm. A semicircle has been added to one side of the square and a quarter of a circle (quadrant) added to another side. Calculate the area of the shape.

Solution

Area of square $= 6^2$

$= 36 \text{ cm}^2$

Radius of semicircle $= 3 \text{ cm}$

Area of semicircle $= \frac{1}{2} \times \pi \times 3^2$

$= 14.1 \text{ cm}^2$ (3 s.f.)

Radius of quarter circle $= 6 \text{ cm}$

Area of quadrant $= \frac{1}{4} \times \pi \times 6^2$

$= 28.3 \text{ cm}^2$ (3 s.f.)

Total area $= 36 + 14.1 + 28.3$

$= 78.4 \text{ cm}^2$ (3 s.f.)

Exercises

1. (a) Calculate the area of each part of the following shape:

(b) What is the total area of the shape?
2. Calculate the area of each of the following shapes:

(a) ![Shape A]

(b) ![Shape B]

(c) ![Shape C]

3. The following diagram shows the plan of a patio. Calculate the area of the patio.

![Patio Diagram]
4. Calculate the area and perimeter of the following shape:

5. A Christmas decoration consists of a disc with two holes cut in it, as shown.
   The disc has radius 3.8 cm.
   The large hole has radius 1.2 cm.
   The small hole has radius 0.2 cm.
   Both sides of the decoration are painted.
   Calculate the area that is painted.

6. Calculate the area and perimeter of the shape shown:

7. A set of steps is to be built with a semicircular shape. Three of the steps are shown in the following diagrams. Calculate the area of each of these three steps.
8. A car wheel has radius 0.25 m. How far does the car travel if the wheel goes round:
   (a) 10 times,
   (b) 600 times?

9. A wheel of a bicycle has diameter 60 cm. How many times does the wheel revolve on a journey of length:
   (a) 500 m,
   (b) 2.6 km?

10. Calculate the area and perimeter of the following shapes:
    (a) 
    (b)
16.6 Volume and Surface Area of a Cylinder

In an earlier unit you will have considered the volume of a triangular prism. The formula used for this can be applied to determine the volume of a cylinder.

Volume of prism  =  area of cross-section × length
                 =  $A l$

A cylinder is a prism with a circular cross-section.
Volume of cylinder  =  $A × h$
                   =  $\pi r^2 h$

The total surface area of the cylinder can be determined by splitting it into 3 parts as below:

The curved surface can be opened out to form a rectangle. The length of one side is equal to the height, $h$, of the cylinder; the other is equal to the circumference of the cross-section, $2\pi r$.

Total area  =  area of curved surface + area of top + area of bottom
            =  $2\pi rh + \pi r^2 + \pi r^2$
            =  $2\pi rh + 2\pi r^2$
Example 1
Calculate the volume and surface area of the cylinder shown in the diagram.

Solution
The radius of the base of the cylinder is 3 cm.
Volume \[ V = \pi r^2 h \]
\[ = \pi \times 3^2 \times 8 \]
\[ = 226 \text{ cm}^3 \text{ (3 s.f.)} \]
Surface area \[ = 2\pi rh + 2\pi r^2 \]
\[ = 2 \times \pi \times 3 \times 8 + 2 \times \pi \times 3^2 \]
\[ = 207 \text{ cm}^2 \text{ (3 s.f.)} \]

Example 2
The diagram shows a sheet of card that is to be used to make the curved surface of a cylinder of height 8 cm.
(a) Calculate the radius of the cylinder.
(b) Use your answer to part (a) to calculate the area of card that would be needed to make ends for the cylinder.
(c) Calculate the volume of the cylinder.

Solution
(a) The circumference of the cross-section is 22 cm, so
\[ 2\pi r = 22 \]
\[ r = \frac{22}{2\pi} \]
\[ = \frac{11}{\pi} \]
\[ = 3.50 \text{ cm (3 s.f.)} \]
(b) Area of ends \[ = 2 \times \pi r^2 \]
\[ = 2 \times \pi \times 3.50^2 \]
\[ = 77.0 \text{ cm}^2 \text{ (3 s.f.)} \]
(b) Volume of cylinder \[ = \pi r^2 h \]
\[ = \pi \times 3.5^2 \times 8 \]
\[ = 308 \text{ cm}^3 \text{ (3 s.f.)} \]
Exercises

1. Calculate the volume of the cylinder shown.

2. Look at the dimensions of the following cylinders:

   (a) Without doing any calculations, decide which cylinder you think has the greatest volume.
   (b) Determine the volume of each cylinder and see if you were correct.

3. Calculate the total surface area of the following cylinder:
4. The following diagrams show two cylinders, A and B:

(a) Show that both cylinders have the same volume.
(b) Calculate the total surface area of each cylinder.

5. A cylinder has volume $250 \text{ cm}^3$ and base radius 6 cm.
(a) Calculate the height of the cylinder.
(b) Calculate the total surface area of the cylinder.

6. A cylinder has volume $300 \text{ cm}^3$ and height 9 cm. Calculate the diameter of the cylinder.

7. The curved surface of a cylinder is to be made from a rectangular sheet of material which is 18 cm by 32 cm.
(a) Explain why two different cylinders could be made from this sheet.
(b) Calculate the radius of each of the cylinders.
(c) Calculate the volume of each cylinder.

8. A cylinder has height 11 cm. The area of the curved surface of the cylinder is $40 \text{ cm}^2$. Calculate the volume of the cylinder.

9. The diagram shows the cross-section of a clay pipe. The length of the pipe is 40 cm. Calculate the volume of clay needed to make the pipe.

10. Calculate the volume and the total surface area of the shape shown.
It is very useful to be able to estimate lengths, masses, etc. because it may not always be easy to measure them. Some useful hints for estimating are listed below:

The height of a standard door is about 2 m.
The length of an adult pace is about 1 m.
The length of a size 8 shoe is about 30 cm.
Most adults are between 1.5 m and 1.8 m in height.
It takes about 15 minutes to walk one kilometre.
The mass of a standard bag of sugar is 1 kg.
The mass of a family car is about 1 tonne.
1 hectare = 10 000 m² (about 2 football pitches).
A teaspoon holds about 5 ml of liquid.
The volume of a normal can of drink is about 330 cm³.

Example 1
The diagram shows a tall man standing beside a factory.

*Estimate:*  
(a) the height of the factory,  
(b) the height of the door.

**Solution**  
(a) The diagram shows that the height of the factory is approximately 5 times the height of the man.  
   Estimate the man's height as 1.8 m.  
   An estimate for the height of the factory is \[5 \times 1.8 \, \text{m} = 9 \, \text{m}\]
(b) The height of the door is approximately $1 \frac{1}{2}$ times the height of the man. An estimate for the height of the door is

$$1 \frac{1}{2} \times 1.8 = 2.7 \text{ m}$$

**Example 2**

The diagram shows a tall person standing behind a lorry.

Estimate the length and height of the lorry, assuming that the height of the person is about 1.8 m.

**Solution**

The diagrams show how to make estimates for the height and length.

Height $\approx 2 \times 1.8 \text{ m}$

$\approx 3.6 \text{ m}$

Length $\approx 3 \frac{1}{2} \times 1.8 \text{ m}$

$\approx 6.3 \text{ m}$

*Note* If the height of the person was actually 1.6 m, the estimates for the height and length would change to 3.2 m and 5.6 m respectively.

**Exercises**

1. *Estimate* the following in your classroom:
   
   (a) length of room,
   
   (b) width of room,
   
   (c) height of room,
   
   (d) height of door,
   
   (e) height of windows,
   
   (f) width of black/white board.
2. *Estimate* the following:
   (a) the height of a football goal,
   (b) the width of a hockey pitch,
   (c) the width of a football goal,
   (d) the height of a netball post.

   Measure the *actual* heights and widths and compare with your estimates.

3. (a) *Estimate* the size of your text book (width, height and thickness).
   (b) *Measure* your text book to see how good your estimates were.

4. *Estimate* the lengths of the following vehicles:
   (a) a car,
   (b) a bus,
   (c) an articulated lorry,
   (d) a motorcycle.

5. Collect together a number of items of various masses.
   (a) Copy and complete the table, writing in the *actual* mass after each estimate.

<table>
<thead>
<tr>
<th>Item</th>
<th>Estimate of Mass</th>
<th>Actual Mass</th>
</tr>
</thead>
<tbody>
<tr>
<td>Text book</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Can of drink</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

   (b) Do you become more accurate at estimating as you have more practice?

6. *Estimate*, in grams or kilograms, the mass of the following:
   (a) a table tennis ball,
   (b) a chair,
   (c) a large dog,
   (d) your school bag, when full,
   (e) a calculator,
   (f) a pen.
7. Estimate, in ml or litres, the volume of milk you would:
   (a) add to a cup of tea,
   (b) pour on to cereal in a bowl,
   (c) pour into a mug.

8. Estimate the volume of:
   (a) a football,
   (b) a tennis ball,
   (c) a table tennis ball,
   (d) a hockey ball.

9. Jo estimates that the height of a double-decker bus is 9 m. Do you think that this is a reasonable estimate? Explain why.

10. Tony estimates that the capacity of a thermos flask is 1 litre, because it is about the same size and shape as a 1 litre lemonade bottle. Explain whether or not you think he has made a good estimate.

11. Which of the following would be the best estimate for the mass of an apple:
    A  1 kg
    B  2 grams
    C  200 grams
    D  20 grams
    E  800 grams

12. Which of the following would be the best estimate for the diameter of a saucer:
    A  16 cm
    B  16 mm
    C  16 m
    D  8 mm
    E  80 cm

13. Which of the following would be the best estimate for the capacity of a tea cup:
    A  15 ml
    B  1500 ml
    C  0.5 litres
    D  5 litres
    E  150 ml
17.2 The Metric System: Conversion Between Units

The metric (decimal) system uses a number of standard prefixes for units of length, mass, etc.

The three most important are:

\[\text{kilo} = 1000\]

\[\text{centi} = \frac{1}{100}\]

\[\text{milli} = \frac{1}{1000}\]

You will have met many of these already, for example,

<table>
<thead>
<tr>
<th>Unit Conversion</th>
<th>Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 millimetre = 1000 metres</td>
<td>so 1 metre = 1000 millimetres</td>
</tr>
<tr>
<td>1 kilogram = 1000 grams</td>
<td>so 1 gram = (\frac{1}{1000}) kilogram</td>
</tr>
<tr>
<td>1 centimetre = (\frac{1}{100}) metre</td>
<td>so 1 metre = 100 centimetres</td>
</tr>
<tr>
<td>1 millilitre = (\frac{1}{1000}) litre</td>
<td>so 1 litre = 1000 millilitres</td>
</tr>
</tbody>
</table>

It is also useful to know that:

\[1 \text{ cm}^3 = 1 \text{ millilitre (ml)}\]

and

\[1000 \text{ kg} = 1 \text{ tonne}\]

Example 1

Complete each of the following statements:

(a) \(150 \text{ cm} = \boxed{\text{m}}\)  
(b) \(360 \text{ mm} = \boxed{\text{m}}\)

(c) \(3.6 \text{ tonnes} = \boxed{\text{kg}}\)  
(d) \(62 \text{ ml} = \boxed{\text{litres}}\)
Solution

(a) $150 \text{ cm} = 150 \times \frac{1}{100} = 1.5 \text{ m}$

(b) $360 \text{ mm} = 360 \times \frac{1}{1000} = 0.36 \text{ m}$

(c) $3.6 \text{ tonnes} = 3.6 \times 1000 = 3600 \text{ kg}$

(d) $62 \text{ ml} = 62 \times \frac{1}{1000} = 0.062 \text{ litres}$

Example 2

John adds 250 ml of water to a jug that already contains 1.2 litres of water. How much water is now in the jug?

Solution

$1.2 \text{ litres} = 1.2 \times 1000$

$= 1200 \text{ ml}$

Total volume $= 1200 + 250$

$= 1450 \text{ ml} \text{ or} \ 1.45 \text{ litres}$

Exercises

1. Change the following lengths into mm:
   (a) 4 cm  (b) 7 cm  (c) 26 cm  (d) 835 cm
   (e) 6.2 cm  (f) 14.7 cm  (g) 9.25 cm  (h) 0.04 cm
   Change the following lengths into cm:
   (i) 60 mm  (j) 80 mm  (k) 340 mm  (l) 9450 mm
   (m) 87 mm  (n) 262 mm  (o) 67.9 mm  (p) 6 mm

2. Change the following lengths into cm:
   (a) 7 m  (b) 18 m  (c) 36 m  (d) 904 m
   (e) 4.3 m  (f) 53.9 m  (g) 28.38 m  (h) 0.09 m
   Change the following lengths into m:
   (i) 800 cm  (j) 500 cm  (k) 760 cm  (l) 2150 cm
   (m) 365 cm  (n) 57 cm  (o) 77.6 cm  (p) 6 cm
3. Change the following lengths into m:
   (a)  5 km  (b)  11 km  (c)  63 km  (d)  423 km
   (e)  7.4 km  (f)  2.56 km  (g)  14.321 km  (h)  0.07 km

   Change the following lengths into km:
   (i)  6000 m  (j)  17 000 m  (k)  53 000 m  (l)  4750 m
   (m)  807 m  (n)  62 m  (o)  3 m  (p)  29.3 m

4. Change the following masses into g:
   (a)  6 kg  (b)  8 kg  (c)  15 kg  (d)  92 kg
   (e)  1.7 kg  (f)  5.47 kg  (g)  2.925 kg  (h)  0.004 kg

   Change the following masses into kg:
   (i)  3000 g  (j)  40 000 g  (k)  8340 g  (l)  29 750 g
   (m)  237 g  (n)  52 g  (o)  9 g  (p)  3.6 g

5. Copy and complete each of the following statements:
   (a)  320 mm  =  
   (b)  6420 mm  =  
   (c)  642 mm  =  
   (d)  888 cm  =  
   (e)  224 cm  =  
   (f)  45 m  =  
   (g)  320 m  =  
   (h)  8.73 m  =  

6. Convert the following masses to kg:
   (a)  8.2 tonnes  (b)  160 tonnes
   (c)  88 g  (d)  3470 g

7. Convert the following masses to g:
   (a)  3.6 kg  (b)  3.7 tonnes
   (c)  840 mg  (d)  62 mg

8. Convert the following volumes to ml:
   (a)  \( \frac{1}{4} \) litre  (b)  22 litres
   (c)  0.75 litres  (d)  450 cm\(^3\)
9. Convert the following volumes to litres:
   (a) 4740 ml  
   (b) 64 ml  
   (c) 300 ml  
   (d) 3600 cm³

10. A cake recipe requires 0.25 kg of flour. Rachel has 550 grams of flour. How much flour will she have left when she has made the cake? Give your answer
   (a) in kg,  
   (b) in g.

11. A chemistry teacher requires 250 mg of a chemical for an experiment. He has 30 grams of the chemical. How many times can he carry out the experiment?

12. A bottle contains 1.5 litres of cola. Hannah drinks 300 ml of the cola and then Ben drinks 450 ml. How much of the cola is left? Give your answer
   (a) in ml,  
   (b) in litres.

13. Emma estimates that the mass of one sweet is 20 grams. How many sweets would you expect to find in a packet that contains 0.36 kg of these sweets?

14. To make a certain solution, 50 grams of a chemical must be dissolved in 4 litres of water.
   (a) How much of the chemical should be dissolved in 1 litre of water?
   (b) How many ml of water would be needed for 200 mg of the chemical?
   (c) How many grams of the chemical would be dissolved in 500 ml of water?

17.3 Estimating Imperial Units of Length, Mass and Capacity

The imperial system was used, until very recently, for all weights and measures throughout the UK. There are many aspects of everyday life where the system is still in common usage. Road signs are an obvious example where miles instead of kilometres are used. In this section we look at estimating in these units; the following list gives some useful facts to help you.
The height of a tall adult is about 6 feet.
The width of an adult thumb is about 1 inch.
The length of a size 8 shoe is about 1 foot.
An adult pace is about 1 yard.
The mass of a bag of sugar is just over 2 pounds.
An old-style bottle of milk contains 1 pint.
It takes about 20 minutes to walk one mile.

You will find the following abbreviations used for imperial units:

<table>
<thead>
<tr>
<th>Abbreviation</th>
<th>Equivalent</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 yard</td>
<td>1 yd</td>
</tr>
<tr>
<td>6 feet</td>
<td>6 ft = 6 '</td>
</tr>
<tr>
<td>9 inches</td>
<td>9 in = 9 &quot;</td>
</tr>
<tr>
<td>8 ounces</td>
<td>8 oz</td>
</tr>
<tr>
<td>7 pounds</td>
<td>7 lb</td>
</tr>
</tbody>
</table>

but be careful not to use m as an abbreviation for miles because m is a standard abbreviation for metres.

Example 1

Estimate the length of the following line, in inches:

Solution

The diagram shows the line itself and the outline of 4 adult thumbs:

So the length can be estimated as 4 inches.

Example 2

The picture shows a man standing next to a wall with a gate in it:

Estimate the height in feet of both the wall and the gate.
Solution

The wall is about $1 \frac{1}{3}$ times the height of the man, so taking the height of the man as 6 feet, gives

$$\text{height of wall} = 1 \frac{1}{3} \times 6 \approx 8 \text{ feet}$$

The gate is about the same height as the man, so its height can be estimated as six feet.

Exercises

1. Estimate the length of each of the lines below, in inches. Then measure each line to check your estimate.
   (a)
   (b)
   (c)
   (d)

2. (a) Estimate the size of the top of your desk, in inches.
   (b) Measure your desk and see how accurate your estimate was.

3. (a) Estimate the heights of 4 of your friends, in feet and inches.
   (b) Measure these friends and see how accurate your estimates were.

4. Estimate the length and width of your classroom, in feet.

5. Estimate the total mass of 3 maths text books, in pounds.

6. Estimate the mass of an apple, in ounces. (Remember that there are 16 ounces in 1 lb.)

7. Estimate the capacity of a mug, in pints.
8. *Estimate* the mass of your shoe, in pounds. Check your estimate if possible.

9. *Estimate* the dimensions of a football or hockey pitch, in yards.

10. A fish tank is in the shape of a cube with sides of length 1 foot. *Estimate* the volume of this tank in pints.

### 17.4 Metric and Imperial Units

As both metric and imperial units are in general use, you need to be able to convert between the two systems. The list below contains a number of useful conversion facts which you will need in the examples and exercises that follow.

<table>
<thead>
<tr>
<th>Metric Unit</th>
<th>Equivalent Imperial Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>8 km</td>
<td>≈ 5 miles</td>
</tr>
<tr>
<td>1 m</td>
<td>≈ 40 inches</td>
</tr>
<tr>
<td>30 cm</td>
<td>≈ 1 foot</td>
</tr>
<tr>
<td>2.5 cm</td>
<td>≈ 1 inch</td>
</tr>
<tr>
<td>1 kg</td>
<td>≈ 2.2 lbs</td>
</tr>
<tr>
<td>1 litre</td>
<td>≈ $1\frac{3}{4}$ pints</td>
</tr>
<tr>
<td>1 gallon</td>
<td>≈ $4\frac{1}{2}$ litres</td>
</tr>
<tr>
<td>1 acre</td>
<td>≈ $2\frac{1}{5}$ hectare</td>
</tr>
<tr>
<td>450 g</td>
<td>≈ 1 lb</td>
</tr>
</tbody>
</table>

The following list reminds you of some of the relationships in the imperial system:

<table>
<thead>
<tr>
<th>Imperial Unit</th>
<th>Equivalent Metric Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 lb</td>
<td>= 16 ounces</td>
</tr>
<tr>
<td>1 stone</td>
<td>= 14 lb</td>
</tr>
<tr>
<td>1 mile</td>
<td>= 1760 yards</td>
</tr>
<tr>
<td>1 yard</td>
<td>= 3 feet</td>
</tr>
<tr>
<td>1 foot</td>
<td>= 12 inches</td>
</tr>
<tr>
<td>1 gallon</td>
<td>= 8 pints</td>
</tr>
<tr>
<td>1 chain</td>
<td>= 22 yards</td>
</tr>
<tr>
<td>1 furlong</td>
<td>= 220 yards</td>
</tr>
</tbody>
</table>
Also note that \[ 1 \text{ acre} = 4840 \text{ square yards} \] (approximately the area of a football pitch).

Conversions between metric and imperial units are not precise, so we always round the converted figure, taking the context into account (see Examples 1 and 2 below).

**Example 1**

While on holiday in France, a family see the following road-sign:

PARIS 342 km

How many *miles* are the family from Paris?

**Solution**

*Note* \[ 8 \text{ km} = 5 \text{ miles} \]

Distance from Paris \[ = 342 \times \frac{5}{8} \text{ miles} \] \[ = 213.75 \text{ miles} \]

The family are therefore about 214 miles from Paris.

**Example 2**

A bottle contains 2.5 litres of milk. How many *pints* of milk does the bottle contain?

**Solution**

*Note* \[ 1 \text{ litre} = \frac{3}{4} \text{ pints} \]

Volume of milk \[ = 2.5 \times 1.75 \text{ pints} \] \[ = 4.375 \text{ pints} \]

The bottle contains almost \( 4 \frac{1}{2} \) pints of milk.

**Example 3**

Vera buys 27 litres of petrol for her car. How many *gallons* of petrol does she buy?
Solution

Note 1 gallon $\approx 4.5$ litres

Quantity of petrol $\approx \frac{27}{4.5}$

$\approx 6$ gallons

Vera buys approximately 6 gallons of petrol.

Exercises

1. Change the following lengths into inches:
   (a) 4 feet  (b) 7 feet  (c) 4 feet 2 inches
   (d) 8 feet 7 inches  (e) 5.5 feet  (f) 2 yards
   (g) 5 yards 2 feet  (h) 1 mile

Change the following lengths into feet or feet and inches:
   (i) 60 inches  (j) 48 inches  (k) 17 inches
   (l) 29 inches  (m) 108 inches  (n) 95 inches
   (o) 240 inches  (p) 6 inches

2. Change the following masses into ounces:
   (a) 7 pounds  (b) 11 pounds  (c) 36 pounds
   (d) 904 pounds  (e) 42 pounds  (f) 5.5 pounds
   (g) 2 stone  (h) 9 stone 12 pounds

Change the following masses into pounds or pounds and ounces:
   (i) 80 ounces  (j) 128 ounces  (k) 56 ounces
   (l) 720 ounces  (m) 36 ounces  (n) 77 ounces
   (o) 8 ounces  (p) 4 ounces

3. Change the following volumes into pints:
   (a) 5 gallons  (b) 11 gallons  (c) 63 gallons
   (d) 412 gallons  (e) 7.5 gallons  (f) $\frac{1}{2}$ gallon
   (g) $3\frac{1}{4}$ gallons  (h) 6.875 gallons
Change the following volumes into gallons or gallons and pints:
(i) 56 pints
(j) 160 pints
(k) 4800 pints
(l) 528 pints
(m) 12 pints
(n) 87 pints
(o) 2 pints
(p) 1884 pints

4. Convert the following distances to cm, where necessary giving your answers to 2 significant figures where necessary:
(a) 6 inches
(b) 8 inches
(c) \(7 \frac{1}{2}\) inches
(d) 8 feet
(e) 4 yards
(f) \(1 \frac{1}{4}\) yards

5. The road-sign shown gives distances in km:
Produce a version of the sign with the equivalent distances given in miles.

6. A recipe requires \(\frac{1}{2}\) lb of flour. What is the equivalent amount of flour in:
(a) grams,
(b) kilograms,
(c) ounces?

7. The capacity of a fuel tank is 30 gallons. What is the capacity of the tank in:
(a) litres,
(b) pints?

8. A cow produces an average of 18 pints of milk each time she is milked. Convert this to litres, giving your answer to 1 decimal place.

9. The mass of a parcel is 4 lb 4 oz. Calculate its mass in kilograms, giving your answer to 1 decimal place.

10. Copy and complete the table shown, which can be used to convert speeds between mph and km/h. Where necessary, express your answers to 3 significant figures.
11. A recipe book provides a table for the conversion between ounces and grams. Copy and complete the table, where necessary giving the values correct to 1 decimal place.

<table>
<thead>
<tr>
<th>Ounces</th>
<th>Grams</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
</tr>
<tr>
<td>50</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>100</td>
</tr>
<tr>
<td>4</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>300</td>
</tr>
<tr>
<td></td>
<td>400</td>
</tr>
</tbody>
</table>

12. (a) Julie calculates the number of metres in 1 mile like this:

\[1760 \times 3 \times 0.3 = 1584\]

Jill calculates the number of metres in 1 mile like this:

\[\frac{8 \times 1000}{5} = 1600\]

Describe how the two methods work and explain why they give different answers.

(b) Show two different ways of converting 20 litres to gallons.

13. The heights of 4 children are measured in feet and inches.

(a) Convert these heights to cm:

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Ben</td>
<td>5 ’ 4 &quot;</td>
</tr>
<tr>
<td>Rachel</td>
<td>5 ’ 8 &quot;</td>
</tr>
<tr>
<td>Emma</td>
<td>4 ’ 7 &quot;</td>
</tr>
<tr>
<td>Hannah</td>
<td>3 ’ 1 ”</td>
</tr>
</tbody>
</table>

(b) Calculate the mean height of the four children,

(i) in cm

(ii) in feet and inches.
17.5 Problems in Context

In this section we look at a variety of problems where the context requires us to deal with more than one type of unit. The units may be only metric, or only imperial, or a mixture of both.

Example 1

A school canteen buys a 1 gallon can of fruit juice. The canteen sells the fruit juice in paper cups that each contain 150 ml of drink. How many cups can be filled?

Solution

\[
1 \text{ gallon} \approx 4.5 \text{ litres} \\
\approx 4500 \text{ ml}
\]

So about \( \frac{4500}{150} = 30 \) cups can be filled from one can.

Example 2

Some students take part in a 20-mile sponsored relay run, where each student runs 3000 m and then another student takes over. If each student runs only once, how many students are needed to complete the run?

Solution

\[
20 \text{ miles} \approx 20 \times \frac{8}{5} \text{ km} \\
\approx 32 \text{ km} \\
\approx 32000 \text{ m}
\]

\[
32000 \div 3000 = 10 \text{ remainder } 2000,
\]

so 11 students are needed to complete the run, but one of them will run only about 2000 m.

Example 3

A technology teacher has a 50-yard roll of glass fibre tape. For a project, each student in the class will need 80 cm of tape. There are 30 students in the class. What length of tape will be left over?
Solution

50 yards $= 50 \times 3$

$= 150$ feet

$= 150 \times 0.3$ m

$= 45$ m

Tape used $= 30 \times 80$ cm

$= 2400$ cm

$= 24$ m

Tape left $\approx 45 - 24$ m

$\approx 21$ m

Exercises

1. A glass holds 50 ml of drink. How many glasses can be filled from:
   (a) a 1 litre bottle,
   (b) a 1 gallon can,
   (c) a 3 pint carton?

2. A sheet of wood measures 4 feet by 8 feet. A teacher cuts up the sheet into smaller pieces that measure 10 cm by 20 cm. How many of these smaller sheets can the teacher make?

3. A baker buys a 25 kg sack of flour. He uses 1 lb of flour for each loaf. How many loaves can he make with 1 sack of flour?

4. How many 125 ml glasses can be filled from a can that contains 2 pints of milk?

5. How many books of width 2.5 cm can be put on a shelf of length 3 feet?

6. A ball of wool contains 75 yards. If 22 m are needed for a knitting pattern, what length of wool is left? Give your answer in:
   (a) metres,
   (b) yards,
   (c) feet and inches.
7. If the average length of a car is 4 m, determine the length of a bumper-to-bumper traffic jam containing 2000 cars, in:
   (a) km,   (b) miles.

8. The length of a domino is 2 inches. A group of children placed dominoes end-to-end to form a line of length 100 m. How many dominoes did they use?

9. The diameter of a bicycle wheel is 28 inches. How many times would the wheel go round as the bicycle moves:
   (a) 550 yards,   (b) 1 mile,   (c) 1 km?

10. The mass of 1 litre of water is 1 kg. Determine the mass of:
    (a) 1 pint of water, in ounces,
    (b) 1 gallon of water, in pounds,
    (c) 50 ml of water, in ounces.
In this section we introduce the idea of speed, considering both *instantaneous speed* and *average speed*.

$$\text{Instantaneous speed} = \text{speed at any instant in time}$$

$$\text{Average speed} = \frac{\text{distance travelled}}{\text{time taken}}$$

If a car travels 100 miles in 2 hours,

$$\text{average speed} = \frac{100}{2} = 50 \text{ mph}$$

The car does not travel at a constant speed of 50 mph; its speed varies during the journey between 0 mph and, perhaps, 70 mph. The speed at any time is called the *instantaneous speed*.

The following table lists units in common use for speed and their abbreviations:

<table>
<thead>
<tr>
<th>Distance</th>
<th>Time</th>
<th>Speed</th>
<th>Abbreviation</th>
</tr>
</thead>
<tbody>
<tr>
<td>mile</td>
<td>hours</td>
<td>miles per hour</td>
<td>mph</td>
</tr>
<tr>
<td>kilometres</td>
<td>hours</td>
<td>kilometres per hour</td>
<td>km/h</td>
</tr>
<tr>
<td>metres</td>
<td>hours</td>
<td>metres per hour</td>
<td>m/h</td>
</tr>
<tr>
<td>metres</td>
<td>seconds</td>
<td>metres per second</td>
<td>m/s</td>
</tr>
<tr>
<td>feet</td>
<td>seconds</td>
<td>feet per second</td>
<td>f.p.s. or ft. per sec.</td>
</tr>
<tr>
<td>centimetres</td>
<td>seconds</td>
<td>centimetres per second</td>
<td>cm/sec or cm/s</td>
</tr>
</tbody>
</table>

**Example 1**

Judith drives from Plymouth to Southampton, a distance of 160 miles, in 4 hours.

She then drives from Southampton to London, a distance of 90 miles, in 1 hour and 30 minutes.

Determine her average speed for each journey.
Solution

Plymouth to Southampton
Average speed $= \frac{160}{4}$
$= 40$ mph

Southampton to London
Time taken $= 1$ hour and 30 minutes
$= 1\frac{1}{2}$ hours or $\frac{3}{2}$ hours

Average speed $= 90 \div \frac{3}{2}$

$= 90 \times \frac{2}{3}$

$= 60$ mph

Example 2
John can type 960 words in 20 minutes.
Calculate his typing speed in:
(a) words per minute,
(b) words per hour.

Solution

(a) Typing speed $= \frac{960}{20}$

$= 48$ words per minute

(b) Typing speed $= 48 \times 60$

$= 2880$ words per hour
Exercises

1. Peter drives 320 miles in 8 hours. Calculate his average speed.

2. Daisy drives from Sheffield to London, a distance of 168 miles, in 4 hours. Calculate her average speed.

3. A snail moves 8 m in 2 hours. Calculate the average speed of the snail in metres per hour.

4. A lorry driver keeps a record of each journey he makes. Calculate the average speed for each journey, using the table below:

<table>
<thead>
<tr>
<th>Start</th>
<th>Finish</th>
<th>Start Time</th>
<th>Finish Time</th>
<th>Distance</th>
</tr>
</thead>
<tbody>
<tr>
<td>Brighton</td>
<td>Norwich</td>
<td>0800</td>
<td>1200</td>
<td>172 miles</td>
</tr>
<tr>
<td>Norwich</td>
<td>Carlisle</td>
<td>1400</td>
<td>1900</td>
<td>280 miles</td>
</tr>
<tr>
<td>Carlisle</td>
<td>Cardiff</td>
<td>1000</td>
<td>1800</td>
<td>300 miles</td>
</tr>
<tr>
<td>Cardiff</td>
<td>Exeter</td>
<td>0700</td>
<td>0930</td>
<td>120 miles</td>
</tr>
<tr>
<td>Exeter</td>
<td>Brighton</td>
<td>1030</td>
<td>1530</td>
<td>175 miles</td>
</tr>
</tbody>
</table>

5. Javinda takes $1 \frac{1}{2}$ hours to drive 30 km in the rush hour. Calculate his average speed in km/h.

6. Rebecca cycles 20 miles on her bike in 2 hours and 30 minutes. Calculate her average speed in mph.

7. Julie can type 50 words in 2 minutes.
   Debbie can type 300 words in 15 minutes.
   Calculate the typing speed of each of the girls in:
   (a) words per minute,
   (b) words per hour.

8. Fatima, Emma and Andy each drive from London to Brighton, a distance of 60 miles. Fatima takes 1 hour, Emma takes 2 hours and Andy takes $1 \frac{1}{2}$ hours. Calculate the average speed for each of the drivers.
9. Eva drives from Edinburgh to Dover in 3 stages:

<table>
<thead>
<tr>
<th>Start Time</th>
<th>Finish Time</th>
<th>Distance</th>
</tr>
</thead>
<tbody>
<tr>
<td>Edinburgh</td>
<td>Leeds</td>
<td>0620</td>
</tr>
<tr>
<td>Leeds</td>
<td>London</td>
<td>1035</td>
</tr>
<tr>
<td>London</td>
<td>Dover</td>
<td>1503</td>
</tr>
</tbody>
</table>

Calculate her average speed for each stage of her journey.

10. Delia drives 220 km in $3\frac{1}{2}$ hours. Calculate her average speed correct to the nearest km/h.

18.2 Calculating Speed, Distance and Time

In this section we extend the ideas of speed to calculating distances and times, using the following formulae:

\[
\begin{align*}
\text{Speed} & = \frac{\text{Distance}}{\text{Time}} \\
\text{Distance} & = \text{Speed} \times \text{Time} \\
\text{Time} & = \frac{\text{Distance}}{\text{Speed}}
\end{align*}
\]

Example 1

Jane drives at an average speed of 45 mph on a journey of 135 miles. How long does the journey take?

Solution

\[
\begin{align*}
\text{Time} & = \frac{\text{distance}}{\text{speed}} \\
& = \frac{135}{45} \\
& = 3 \text{ hours}
\end{align*}
\]
Example 2

Chris cycles at an average speed of 8 mph. If he cycles for $6\frac{1}{2}$ hours, how far does he travel?

**Solution**

Distance = speed $\times$ time

$$= 8 \times 6\frac{1}{2}$$

$$= 52 \text{ miles}$$

Example 3

Nikki has to travel a total of 351 miles. She travels the first 216 miles in 4 hours.

(a) Calculate her average speed for the first part of the journey.

(b) If her average speed remains the same, calculate the total time for the complete journey.

**Solution**

(a) Average speed = $\frac{\text{distance}}{\text{time}}$

$$= \frac{216}{4}$$

$$= 54 \text{ mph}$$

(b) Time = $\frac{\text{distance}}{\text{speed}}$

$$= \frac{351}{54}$$

$$= 6.5 \text{ hours}$$
Exercises

1. Calculate the distance that you would travel if you drove for:
   (a) 3 hours at 20 mph  (b) 8 hours at 60 mph
   (c) \( \frac{1}{2} \) hour at 76 mph  (d) \( 1 \frac{1}{2} \) hours at 42 mph
   (e) \( 6 \frac{1}{4} \) hours at 40 mph  (f) 30 minutes at 33 mph
   (g) 45 minutes at 60 mph  (h) 90 minutes at 45 mph

2. How long does it take to travel:
   (a) 120 miles at 40 mph  (b) 300 miles at 50 mph
   (c) 240 miles at 60 mph  (d) 385 miles at 70 mph
   (e) 60 miles at 40 mph  (f) 360 miles at 30 mph
   (g) 390 miles at 60 mph  (h) 253 miles at 46 mph

3. A car travels 300 miles in 5 hours. Calculate the average speed of the car in:
   (a) mph,
   (b) miles per minute.
   How long does it take for the car to travel 82 miles?

4. Janet and Bill leave their home at the same time. Janet has 60 miles to travel and drives at 40 mph. Bill has 80 miles to travel and also drives at 40 mph.
   (a) How long does Janet's journey take?
   (b) How much longer does Bill spend driving than Janet?

5. An athlete can run long distances at 4 metres per second. How far can she run in:
   (a) 50 seconds,
   (b) 3 minutes,
   (c) 1 hour,
   (d) \( 2 \frac{1}{2} \) hours?

6. Andrew rows at an average speed of 2 metres per second.
   (a) How long does it take him to row:
      (i) 70 m,  (ii) 800 m,  (iii) \( 1 \frac{1}{2} \) km?
(b) How far can Andrew row in:
   (i) 12 seconds,  (ii) $3\frac{1}{2}$ minutes,  (iii) 4 hours?

7. A snail moves 5 m in 2 hours. If the snail moves at the same speed, calculate:
   (a) the time it takes to move 20 m,
   (b) the distance it would move in $3\frac{1}{2}$ hours,
   (c) the time it takes to move 1 m,
   (d) the distance that it moves in 15 minutes.

8. Laura drives for 3 hours at 44 mph.
   Clare drives 144 miles in 4 hours.
   (a) Who travels the greater distance?
   (b) Whose speed is the slower?
   (c) How far would Laura travel if she drove for 3 hours at the same speed as Clare?

9. A lorry travels for 3 hours at 48 mph and then for 2 hours at 53 mph.
   (a) What is the total distance travelled by the lorry?
   (b) What is the average speed for the whole journey?

10. Sally drives for $2\frac{1}{2}$ hours at 50 mph, then drives 80 miles at 40 mph, and finally drives for 30 minutes at 60 mph.
    (a) Calculate the total distance that Sally drives.
    (b) Calculate the time that Sally takes for the journey.
    (c) Calculate her average speed for the whole journey.

18.3 Problems with Mixed Units

In this section we consider working with mixed units, and with changing units used for speeds.

Example 1
   (a) Convert 1 hour 24 minutes to hours (decimal).
   (b) Write 2.32 hours in hours and minutes.
18.3

Solution

(a) \( \frac{24}{60} = 0.4 \)

Therefore,

\[ 1 \text{ hr 24 mins} = 1.4 \text{ hours} \]

(b) \( 0.32 \times 60 = 19.2 \)

Therefore,

\[ 2.32 \text{ hours} = 2 \text{ hrs 19.2 mins} \]

Example 2
A car travels 200 miles in 3 hours and 20 minutes. Calculate the average speed of the car in mph.

Solution

3 hours 20 minutes = \( \frac{3 \times 20}{60} \)

= \( 3 \frac{1}{3} \) hours

Speed = distance ÷ time

= \( 200 ÷ 3 \frac{1}{3} \)

= \( 200 ÷ \frac{10}{3} \)

= \( 200 \times \frac{3}{10} \)

= 60 mph

Example 3
An athlete runs 1500 m in 3 minutes and 12 seconds. Calculate the average speed of the athlete in m/s.

Solution

3 minutes 12 seconds = \( 3 \times 60 + 12 \)

= 192 seconds
Speed \( = \frac{\text{distance}}{\text{time}} \)

\( = \frac{1500}{192} \)

\( = 7.8 \text{ m/s to 1 decimal place} \)

Example 4
A bus travels at a speed of 40 km/h. Calculate the speed of the bus in:
(a) m/s
(b) mph.

Solution
(a) \( 1 \text{ km} = 1000 \text{ m} \)

\( 40 \text{ km/h} = 1000 \times 40 \text{ m/hr} \)

1 hour = \( 60 \times 60 \)

= 3600 seconds

\( 40 \text{ km/h} = \frac{1000 \times 40}{3600} \)

\( = 11.1 \text{ m/s to 1 decimal place} \)

(b) \( 1 \text{ km} = \frac{5}{8} \text{ mile} \)

So \( 40 \text{ km/h} = \frac{5}{8} \times 40 \)

\( = 25 \text{ mph} \)

Example 5
Convert a speed of 8 m/s to mph.

Solution
\( 8 \text{ m/s} = 8 \times 3600 \text{ m/h} \)

\( = 28800 \text{ m/h} \)

\( = 28.8 \text{ km/h} \)

\( 28.8 \times \frac{5}{8} = 18 \text{ mph} \)
Exercises

1. Convert the following times from hours and minutes to hours, giving your answers as mixed numbers and decimals, correct to 2 decimal places.
   (a) 1 hour 40 minutes  
   (b) 3 hours 10 minutes  
   (c) 1 hour 6 minutes  
   (d) 2 hours 18 minutes  
   (e) 3 hours 5 minutes  
   (f) 6 hours 2 minutes  
   (g) 1 hour 7 minutes  
   (h) 2 hours 23 minutes

2. Change the following times to hours and minutes:
   (a) $1 \frac{1}{4}$ hours  
   (b) 1.2 hours  
   (c) 3.7 hours  
   (d) 4.4 hours  
   (e) 1.45 hours  
   (f) 3.65 hours

3. A car travels 60 miles in 50 minutes. Calculate the average speed of the car in mph.

4. Jane drives 80 miles in 1 hour and 40 minutes. Calculate her average speed.

5. Convert the following speeds to km/h:
   (a) 60 mph  
   (b) 43 m/s  
   (c) 66 m/s  
   (d) 84 mph

6. Convert the following speeds to mph:
   (a) 16 m/s  
   (b) 82 km/h  
   (c) 48 km/h  
   (d) 7 m/s

7. Alec drives 162 km in 2 hours and 12 minutes. Calculate his average speed in:
   (a) km/h  
   (b) m/s  
   (c) mph
   Give your answers to 2 decimal places.

8. Jai drives 297 miles in 5 hours and 24 minutes.
   (a) Calculate his average speed in mph.
   (b) He then drives for a further 1 hour and 28 minutes at the same average speed. How far has he travelled altogether?
   Give your answers to 2 decimal places.
9. A train travels at 40 m/s. Calculate the time it takes to travel:
   (a) 30 000 m,
   (b) 50 km,
   (c) 200 miles.

10. A long distance runner runs at an average speed of 7 mph. How long will it take the runner to run:
   (a) 20 miles,
   (b) 15 km,
   (c) 10 000 m?

18.4 Distance-Time Graphs

Graphs that show distance against time can be used to describe journeys. The vertical scale shows the distance from the starting point or reference point.

The graph above illustrates 3 parts of a journey.

The gradient of a straight line gives the speed of the moving object. Gradient is a measure of the speed.

Note that a negative gradient indicates that the object is moving towards the starting point rather than away from it.
Example 1

The graph shows how far a child is from home.

(a) Describe how the child moves.
(b) Calculate the speed of the child on each part of the journey.

Solution

(a) The first part of the graph shows the child moving away from home at a constant speed.

The second (horizontal) part of the graph shows that the child remains in the same position.

The third part of the graph shows the child returning to the starting point at a steady speed.

(b) During the first stage the child travels 1000 m in 80 seconds.

\[
\text{Speed} = \frac{\text{distance}}{\text{time}}
\]

\[
= \frac{1000}{80}
\]

\[
= 12.5 \text{ m/s}
\]

During the second stage the speed of the child is zero.
During the third stage as the child returns, he travels 1000 m in 100 seconds.

\[
\text{Speed} = \frac{\text{distance}}{\text{time}} = \frac{1000}{100} = 10 \text{ m/s}
\]

Example 2

On a journey, Rebecca drives at 50 mph for 2 hours, rests for 1 hour and then drives another 70 miles in \(1 \frac{1}{2}\) hours.

Draw a distance-time graph to illustrate this journey.

**Solution**

First stage

- Travels 100 miles in 2 hours.

Second stage

- Rests, so distance does not change.

Third stage

- Travels 70 miles in \(1 \frac{1}{2}\) hours.
Example 3

The graph shows how Tom's distance from home varies with time, when he visits Ian.

(a) How long does Tom spend at Ian's?
(b) How far is it from Tom's home to Ian's?
(c) For how long does Tom stop on the way to Ian's?
(d) On which part of the journey does Tom travel the fastest?
(e) How fast does Tom walk on the way back from Ian's?

Solution

(a) The longer horizontal part of the graph represents the time that Tom is at Ian's.

\[
\text{Time} = 90 - 40 = 50 \text{ minutes}
\]

(b) 3000 m

(c) Tom stops for 10 minutes, represented by the smaller horizontal part on the graph.

(d) He travels fastest on the second part of the journey to Ian's. This is where the graph is steepest. He travels 2000 m in 10 minutes.

\[
\text{Speed} = \frac{2000}{10} = 200 \text{ m/minute}
\]

\[
= \frac{200 \times 60}{1000} = 12 \text{ km/h}
\]
(e) Tom travels 3000 m in 30 mins.

\[
\text{Speed} = \frac{\text{distance}}{\text{time}}
\]

\[
= \frac{3000}{30}
\]

\[
= 100 \text{ m/minute}
\]

Exercises

1. The following graph illustrates how Jamil moves as he goes to the paper shop:

(a) How long does it take Jamil to cycle to the shop?
(b) What distance does Jamil cycle to get to the shop?
(c) Calculate the speed at which Jamil cycles to the shop.
(d) How long does Jamil spend at the shop?
(e) Calculate the speed at which Jamil cycles on his way home.

2. On a journey, Vera
   ● drives 200 miles in 4 hours
   ● rests for 1 hour
   ● drives another 100 miles in 2 hours.

Draw a distance-time graph for Vera's journey.
3. Describe the 5 parts of the journey (labelled (a), (b), (c), (d) and (e)) represented by the following distance-time graph:

![Distance-time graph]

4. Ray walks 420 m from his house to a shop in 7 minutes. He spends 5 minutes at the shop and then walks home in 6 minutes.
   (a) Draw a distance-time graph for Ray's shopping trip.
   (b) Calculate the speed at which Ray walks on each part of the journey.

5. Mary sprints 200 m in 30 seconds, rests for 45 seconds and then walks back in \(1 \frac{1}{2}\) minutes to where she started the race.
   (a) Draw a distance-time graph for Mary.
   (b) Calculate the speed at which Mary runs.
   (c) Calculate the speed at which Mary walks.

6. After morning school, Mike walks home from school to have his lunch. The distance-time graph on the next page describes his journey on one day, showing his distance from home.
   (a) How far is Mike's home from school?
   (b) How long does it take Mike to walk home?
   (c) At what speed does he walk on the way home? Give your answer in m/s.
   (d) How long does Mike spend at home?
(e) At what speed does he walk back to school? Give your answer in m/s.

7. Helen cycles for 20 minutes at 5 m/s and then for a further 10 minutes at 4 m/s.
   (a) How far does she cycle altogether?
   (b) Draw a distance-time graph for her ride.

8. The distance-time graph shown is for a 3000 m cross-country race, run by Rachel and James.
(a) Describe how James runs the race.
(b) Describe how Rachel runs the race.
(c) When, and how far from the start, does James catch up with Rachel?
(d) Calculate the speed at which James runs.
(e) Calculate the different speeds at which Rachel runs.
(f) Who wins the race?

9. Josh completes a 10 000 m race. He runs the first 2000 m at 5 m/s, the next 7400 m at 4 m/s and the last 600 m at 6 m/s.
(a) Draw a distance-time graph for Josh’s race.
(b) How long does he take to complete the race?

10. Emma runs a 2000 m race. She runs at 5 m/s for the first part of the race and at 4 m/s for the rest of the race. She complete the race in 440 seconds.
(a) Draw a distance-time graph for Emma’s race.
(b) How far does she run at each speed?

11. Describe the journey shown in each of the following graphs:
(a) (b)
(c) (d)
18.5 Other Compound Measures

In the section so far we have considered speed in several different contexts. We will now look at other things such as goals per game, and postage rates.

Example 1

In a football season, Ivor Boot scores 27 goals in 40 matches. Calculate his average scoring rate in goals per match, goals per minute and goals per hour.

Solution

\[
\text{Scoring rate} = \frac{27}{40}
\]

= 0.675 goals per match

\[
\text{Scoring rate} = \frac{27}{40 \times 90} \quad \text{(there are 90 minutes per match)}
\]

= 0.0075 goals per minute

\[
\text{Scoring rate} = \frac{27}{40 \times 1\frac{1}{2}}
\]

= 0.45 goals per hour

Example 2

A package has a mass of 200 grams. It can be posted first class for 60p, or second class for 47p. Calculate the cost per gram for first and second class post.

Solution

\[
\text{First Class} \quad \text{Cost per gram} = \frac{60}{200}
\]

= 0.3p

\[
\text{Second Class} \quad \text{Cost per gram} = \frac{47}{200}
\]

= 0.235p
Exercises

1. Three boys play football for a school team. The numbers of goals scored and matches played are listed below:

<table>
<thead>
<tr>
<th>Number of Goals Scored</th>
<th>Number of Matches Played</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ian</td>
<td>16</td>
</tr>
<tr>
<td>Ben</td>
<td>22</td>
</tr>
<tr>
<td>Sergio</td>
<td>9</td>
</tr>
</tbody>
</table>

(a) Who scores the most goals per match?
(b) Who scores the least goals per match?

2. Alison plays 20 games for her school hockey team and scores 18 goals. Each match lasts 90 minutes.

Calculate her scoring rate in:
(a) goals per hour,
(b) goals per minute,
(c) goals per match.

3. When playing football, Jai claims to be able, on average, to score a goal every 40 minutes. How many goals would you expect him to score in:
(a) 90 minutes,
(b) 1 hour,
(c) 5 matches,
(d) 40 matches?

4. It costs 96p to send an air mail letter of mass 40 grams to Africa, and 107p to send it to China.

(a) Calculate the cost per gram for each destination.
(b) If the same rates apply to a 50 gram letter, calculate the cost for each destination.

5. A package of mass 80 grams costs 39p to post first class and 31p to post second class. Calculate the cost per gram for first and second class post.

6. A taxi driver charges £3.20 for a 4 km journey. How much does he charge:
(a) per km,
(b) per metre?
7. A taxi service makes a fixed charge of £1.20 and then 78p per km. Calculate the cost for journeys of the following lengths:
   (a) 1 km
   (b) 2 km
   (c) 4.5 km
   (d) 10.5 km

8. Alexi buys a 20 m length of fabric for £18.60.
   (a) What is the cost per m of the fabric?
   (b) What would be the cost of 9.2 m of the fabric?

9. Five people work in a shop. The following table lists the hours worked and the total paid in one week:

<table>
<thead>
<tr>
<th>Hours Worked</th>
<th>Total Paid</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dee</td>
<td>£28.64</td>
</tr>
<tr>
<td>Nadina</td>
<td>£43.44</td>
</tr>
<tr>
<td>Lisa</td>
<td>£302.40</td>
</tr>
<tr>
<td>Mary</td>
<td>£136.80</td>
</tr>
<tr>
<td>Clare</td>
<td>£134.40</td>
</tr>
</tbody>
</table>

   (a) Who is paid the most per hour?
   (b) Who is paid the least per hour?

10. A 5 litre tin of paint is used to paint a wall that measures 6.25 m by 4 m. Calculate the rate at which paint is applied to the wall, in:

   (a) litres per m²,
   (b) cm³ per m²,
   (c) ml per cm².
19

Similarity

19.1 Enlargement

An enlargement increases or decreases the size of a shape by a multiplier known as the scale factor. The angles in the shape will not be changed by the enlargement.

Example 1

Which of the triangles below are enlargements of the triangle marked A? State the scale factor of each of these enlargements.

Solution

B is an enlargement of A, since all the lengths are doubled.
The scale factor of the enlargement is 2.

C is not an enlargement of A.

D is an enlargement of A, since all the lengths are halved.
The scale factor of the enlargement is \( \frac{1}{2} \).

E is not an enlargement of A.

F is an enlargement, since all the lengths are trebled.
The scale factor of the enlargement is 3.
Example 2

Ameer has started to draw an enlargement of the quadrilateral marked A. Copy and complete the enlargement.

Solution

The diagram shows the completed enlargement.

All the lengths have been increased by a factor of $\frac{1}{2}$.

We say that the *scale factor* of the enlargement is $\frac{1}{2}$. 
Exercises

1. Which of the following shapes are enlargements of shape A? State the scale factor of each of these enlargements.

2. Which of the following triangles are not enlargements of the triangle marked A?

3. The diagram below shows four enlargements of rectangle A. State the scale factor of each enlargement.

4. Which two signs below are not enlargements of sign A?
5. Which two of the leaves shown below are enlargements of leaf A?

A  B  C  D  E

6. Which of the flags below are enlargements of flag A?

A  B  C  D  E  F

7. Draw enlargements of the rectangle shown with scale factors:
   (a) 2  (b) 4
   (c) $\frac{1}{2}$  (d) 3

8. Draw enlargements of the triangle shown with scale factors:
   (a) 2
   (b) 3
   (c) $\frac{1}{2}$
9. Denise has started to draw an enlargement of the shape below. Copy and complete her enlargement.

10. Kristian has started to draw an enlargement of the shape below. Copy and complete his enlargement.

19.2 Similar Shapes

Similar shapes are those which are enlargements of each other; for example, the three triangles shown below are similar:
It is possible to calculate the lengths of the sides of similar shapes.

**Example 1**

The following diagram shows two similar triangles:

![Diagram of two similar triangles](image)

Calculate the lengths of the sides \( BC \) and \( DF \).

**Solution**

Comparing the sides \( AB \) and \( DE \) gives:

\[
AB = 4 \times DE
\]

So, all the lengths in the triangle \( ABC \) will be 4 times the lengths of the sides in the triangle \( DEF \).

\[
BC = 4 \times EF
\]

\[
= 4 \times 3
\]

\[
= 12 \text{ cm}
\]

\[
AC = 4 \times DF
\]

\[
10 = 4 \times DF
\]

\[
DF = \frac{10}{4}
\]

\[
= 2.5 \text{ cm}
\]

**Example 2**

The following diagram shows 2 similar triangles:

![Diagram of two similar triangles](image)
Calculate the lengths of the sides $AC$ and $DE$.

**Solution**

Comparing the lengths $BC$ and $EF$ gives:

$$EF = 2.5 \times BC$$

So the lengths in the triangle $DEF$ are 2.5 times longer than the lengths in the triangle $ABC$.

$$DE = 2.5 \times AB = 2.5 \times 5 = 12.5 \text{ cm}$$

$$DF = 2.5 \times AC = 7.5 = 2.5 \times AC$$

$$AC = \frac{7.5}{2.5} = 3 \text{ cm}$$

**Example 3**

In the following diagram, the sides $AE$ and $BC$ are parallel.

(a) Explain why $ADE$ and $CDB$ are similar triangles.

(b) Calculate the lengths $DE$ and $CD$.

**Solution**

(a) $\angle ADE$ and $\angle CDB$ are opposite angles and so are equal.

Because $AE$ and $BC$ are parallel, $\angle DBC = \angle DEA$ and $\angle EAD = \angle BCD$. 

136
As the triangles have angles the same size, they must be similar.

(b) Comparing $\text{A}\text{E}$ and $\text{B}\text{C}$ shows that the lengths in the larger triangle are 3 times the lengths of the sides in the smaller triangle, so

\[\text{D}\text{C} = 3 \times \text{A}\text{D}\]
\[= 3 \times 3\]
\[= 9 \text{ cm}\]

and

\[\text{B}\text{D} = 3 \times \text{D}\text{E}\]
\[12 = 3 \times \text{D}\text{E}\]
\[\text{D}\text{E} = \frac{12}{3}\]
\[= 4 \text{ cm}\]

**Exercises**

1. The following diagram shows two similar rectangles:

Determine the length of the side $\text{C}\text{D}$. 
2. The following diagram shows two similar triangles:

Calculate the lengths of:
(a) A B  
(b) E F

3. Two similar isosceles triangles are shown in the diagram below:

(a) What is the length of D E ?
(b) What is the length of A C ?
(c) Calculate the length of B C.

4. The following diagram shows two similar triangles:

Calculate the lengths of the sides G E and F G.
5. The following diagram shows three similar triangles:

![Diagram of three similar triangles]

Calculate the length of:
(a) $E G$
(b) $H J$
(c) $E F$
(d) $A B$

6. The following diagram shows 3 similar triangles:

![Diagram of three similar triangles]
7. The following diagram shows two similar shapes:

![Diagram of similar shapes]

The length of the side \( AB \) is 6 cm and the length of the side \( IJ \) is 4 cm.

(a) If \( AH = 12 \text{ cm} \), calculate the length \( IP \).

(b) If \( BC = 3 \text{ cm} \), calculate the length \( JK \).

(c) If \( DE = BC \), determine the length \( LM \).

(d) Calculate the lengths \( FG \) and \( NO \).

(e) If \( MN = 3 \text{ cm} \), determine the length \( EF \).

8. In the diagram below, the lines \( AE \) and \( CD \) are parallel.

![Diagram of parallel lines]

(a) Copy and complete the following statements:

\[
\angle ABE = \angle \]

\[
\angle BAE = \angle \]

\[
\angle AEB = \angle \]

(b) Calculate the lengths of \( AB \) and \( BE \).
9. In the diagram shown below the lines BE and CD are parallel.

(a) Explain why the triangles ABE and ACD are similar.
(b) If the length of AB is 4.4 cm, calculate the lengths of AC and BC.
(c) If the length of AD is 13.5 cm, determine the lengths of AE and DE.

10. In the diagram shown, the lines AB, GD and FE are parallel.

(a) If the length of CE is 15 cm, calculate the lengths of AC, CD and DE.
(b) If the length of BC is 10.8 cm, calculate the length of FG.
19.3 Line, Area and Volume Ratios

In this section we consider what happens to the area and volume of shapes when they are enlarged.

Example 1

The rectangle shown is enlarged with scale factor 2 and scale factor 3.

What happens to the area for each scale factor?

Solution

The area of the original rectangle is

\[
\text{area} = 5 \times 2 = 10 \text{ cm}^2
\]

For an enlargement scale factor 2, the rectangle becomes:

\[
\text{area} = 10 \times 4 = 40 \text{ cm}^2
\]

The area has increased by a factor of 4, or \(2^2\).

For an enlargement scale factor 3, the rectangle becomes:

\[
\text{area} = 15 \times 6 = 90 \text{ cm}^2
\]

The area has increased by a factor of 9, or \(3^2\).
Note
If a shape is enlarged with scale factor $k$, its area is increased by a factor $k^2$.

Example 2
A hexagon has area 60 cm$^2$.
Calculate the area of the hexagon, if it is enlarged with scale factor:
(a) 2  (b) 4  (c) 10

Solution
In each case the area will increase by the scale factor squared.
(a) New area $= 2^2 \times 60$
    $= 4 \times 60$
    $= 240$ cm$^2$
(b) New area $= 4^2 \times 60$
    $= 16 \times 60$
    $= 960$ cm$^2$
(c) New area $= 10^2 \times 60$
    $= 100 \times 60$
    $= 6000$ cm$^2$

Example 3
A cuboid has sides of lengths 3 cm, 4 cm and 5 cm.

Calculate the volume of the cuboid, if it is enlarged with scale factor:
(a) 2  (b) 10
Solution

(a) The dimensions of the cuboid now become, 6 cm, 8 cm and 10 cm.

\[
\text{New volume} = 6 \times 8 \times 10
\]
\[
= 480 \text{ cm}^3
\]

Note that the volume of the original cuboid was 60 cm\(^3\), so the volume has increased by a factor of 8, or 2\(^3\).

(b) The dimensions of the cuboid now become, 30 cm, 40 cm and 50 cm.

\[
\text{New volume} = 30 \times 40 \times 50
\]
\[
= 60 000 \text{ cm}^3
\]

Note that this is 1000, or 10\(^3\), times bigger than the volume of the original cuboid.

\[\text{Note}\]
If a solid is enlarged with scale factor \( k \), its volume is increased by a factor \( k^3 \).
Example 4

A sphere has a volume of 20 cm$^3$. A second sphere has 4 times the radius of the first sphere. Calculate the volume of the second sphere.

Solution

The radius is increased by a factor of 4.

The volume will be increased by a factor of $4^3$.

\[
\text{Volume} = 20 \times 4^3
\]
\[
= 20 \times 64
\]
\[
= 1280 \text{ cm}^3
\]

Exercises

1. Two rectangles are shown below:

(a) Calculate the area of each rectangle.

(b) How many times longer are the sides in rectangle B than those in rectangle A?

(c) How many times bigger is the area of rectangle B?

2. Calculate the area of the rectangle shown if it is enlarged with a scale factor of:

(a) 2  
(b) 3

(c) 6  
(d) 10
3. The following table gives information about enlargements of the triangle shown, which has an area of 6 cm². Copy and complete the table.

<table>
<thead>
<tr>
<th>Length of Sides</th>
<th>Scale Factor</th>
<th>Area</th>
<th>Area Factor</th>
</tr>
</thead>
<tbody>
<tr>
<td>Base</td>
<td>Height</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3 cm</td>
<td>4 cm</td>
<td>1</td>
<td>6 cm²</td>
</tr>
<tr>
<td>12 cm</td>
<td></td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>16 cm</td>
<td></td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>15 cm</td>
<td></td>
<td>6</td>
<td></td>
</tr>
<tr>
<td>30 cm</td>
<td>40 cm</td>
<td>100</td>
<td>600 cm²</td>
</tr>
<tr>
<td>4.5 cm</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

4. The parallelogram shown has an area of 42 cm².

The parallelogram is enlarged with a scale factor of 5. Calculate the area of the enlarged parallelogram.

5. The area of a circle is 50 cm². A second circle has a radius that is 3 times the radius of the first circle. What is the area of this circle?

6. Two similar rectangles have areas of 30 cm² and 480 cm². Describe how the length and width of the two rectangles compare.
7. (a) Determine the volume of each of the following cuboids:

(b) The larger cuboid is an enlargement of the smaller cuboid. What is the scale factor of the enlargement?

(c) How many of the smaller cuboids can be fitted into the larger cuboid?

(d) How many times greater is the volume of the larger cuboid than the volume of the smaller cuboid?

8. A cuboid has dimensions as shown in the diagram.

The cuboid is enlarged to give larger cuboids. Copy and complete the following table:

<table>
<thead>
<tr>
<th>Dimensions</th>
<th>Scale Factor</th>
<th>Volume</th>
<th>Volume Factor</th>
</tr>
</thead>
<tbody>
<tr>
<td>Width</td>
<td>Length</td>
<td>Height</td>
<td>Factor</td>
</tr>
<tr>
<td>3 cm</td>
<td>6 cm</td>
<td>2 cm</td>
<td>1</td>
</tr>
<tr>
<td>6 cm</td>
<td></td>
<td></td>
<td>2</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>4</td>
</tr>
<tr>
<td></td>
<td></td>
<td>10 cm</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>30 cm</td>
<td></td>
</tr>
</tbody>
</table>

9. A tank has a volume of 32 m³. It is enlarged with scale factor 3. What is the volume of the enlarged tank?

10. A cylinder has height 10 cm and volume 42 cm³. An enlargement of the cylinder has height 25 cm. Calculate the volume of the enlarged cylinder.
19.4 Maps and Scale Models

The ideas of how areas and volumes change with enlargement were considered in section 19.3. Here we apply these ideas to maps and scale models.

If a map has a scale $1 : n$, then:
- lengths have a scale of $1 : n$
- areas have a scale of $1 : n^2$.

If a model has a scale of $1 : n$, then:
- lengths have a scale of $1 : n$
- areas have a scale of $1 : n^2$
- volumes have a scale of $1 : n^3$.

**Note on units**

<table>
<thead>
<tr>
<th>Unit</th>
<th>Conversion</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 km</td>
<td>$= 1000 \text{ m}$</td>
</tr>
<tr>
<td></td>
<td>$= 100 000 \text{ cm}$</td>
</tr>
<tr>
<td>1 m²</td>
<td>$= 10 000 \text{ cm}^2$</td>
</tr>
<tr>
<td>1 m³</td>
<td>$= 1 000 000 \text{ cm}^3$</td>
</tr>
</tbody>
</table>

**Example 1**

On a map with a scale of $1 : 20 000$, a garden has an area of $5 \text{ cm}^2$. Calculate the actual area of the garden.

**Solution**

Actual area $= 5 \times 20 000^2$

$= 20 000 000 \text{ cm}^2$

$= 200 000 \text{ m}^2$ (dividing by 10 000)

$= 0.2 \text{ km}^2$ (dividing by 1 000 000)

**Example 2**

A map has a scale of $1 : 500$. A small public garden on the map has an area of $14 \text{ cm}^2$. Calculate the actual area of this garden.

**Solution**

Actual area $= 14 \times 500^2$

$= 3 500 000 \text{ cm}^2$

$= 350 \text{ m}^2$
Example 3

A model car is made on a scale of 1 : 20.
The length of the model is 24 cm.
The area of the windscreen of the model is 32 cm².
The volume of the boot of the model is 90 cm³.

Calculate the actual:
(a) length of the car,
(b) area of the windscreen,
(c) volume of the boot.

Solution

(a) Actual length = 24 × 20
= 480 cm
= 4.8 m

(b) Actual area = 32 × 20²
= 12800 cm²
= 1.28 m²

(c) Actual volume = 90 × 20³
= 720000 cm³
= 0.72 m³

Exercises

1. A model boat is made to a scale of 1 : 10.
The length of the model is 40 cm.
The area of the hull of the model is 500 cm².
The volume of the hull of the model is 3200 cm³.

Calculate the actual:
(a) length of the boat,
(b) area of the hull,
(c) volume of the hull.
2. A map has a scale of 1 : 50 000. On the map the area of a lake is 50 cm². Calculate the actual area of the lake in:
   (a) cm²  (b) m²  (c) km²

3. A model of a tower block is made with a scale of 1 : 60. The volume of the model is 36 000 cm³. Calculate the volume of the actual tower block in m³.

4. A plot of land is represented on a map by a rectangle 2 cm by 5 cm. The scale of the map is 1 : 40 000. Calculate the area of the plot of land in:
   (a) cm²  (b) m²  (c) km²

5. A model of a house is made to a scale of 1 : 30. The height of the model is 20 cm. The area of the roof of the model is 850 cm². The volume of the model house is 144 400 cm³. Calculate the actual:
   (a) height of the house in m,
   (b) area of the roof in m²,
   (c) volume of the house in m³.

6. An aeroplane has a wingspan of 12 m. A model of this plane has a wingspan of 60 cm.
   (a) Calculate the scale of the model.
   (b) The volume of the model is 3015 cm³. Calculate the volume of the actual aeroplane, in m³.
   (c) A badge on the model has area 2 cm². Calculate the area of the actual badge, in cm² and m².

7. A forest has an area of 4 cm² on a map with a scale of 1 : 200 000. Calculate the actual area of the forest, in km².

8. An estate has an area of 50 km². What would be the area of the estate on a map with a scale of 1 : 40 000?
9. An indoor sports stadium has 5000 seats surrounding a playing area with an area of 384 m². The total volume of the stadium is 3840 m³. A model is made to a scale of 1 : 80.

(a) How many seats are there in the model?

(b) What is the area of the playing surface in the model, in cm²?

(c) What is the volume of the model, in cm³?

10. A lake has an area of 5 km². On a map it is represented by an area of 20 cm². What is the scale of the map?
Questionnaires must be designed carefully so that the answers given produce the required information: they should do so without influencing the people completing them. The following list contains points to consider when designing a questionnaire:

- The questions should be worded to provide the information needed by the researcher.
- Care must be taken not to invade people's privacy, so questions that do not relate to the purpose of the questionnaire must be excluded.
- The questions should be capable of being answered reasonably quickly.
- The questions should be easy to understand and should not be ambiguous.
- The questionnaire should not contain biased or leading questions.
- Questions may have possible responses presented in a multiple choice, or YES/NO format.
- Where responses are provided, they should cover every possible answer.
- The responses provided should not overlap.
- The responses provided should not force people to answer in a way that they do not wish to answer.
- The questionnaire should be designed so that the results are easy to analyse.

You should try out a questionnaire in a pilot study of a few people before using it with a large group of people. This will allow you the opportunity to alter questions that do not work well; for example, where they are misinterpreted.
Example 1

Aisha wants to identify the favourite colours of children of different ages. Comment on the following questions that she has decided to ask:

1. *Which age range are you in?*
   - 0 - 5
   - 6 - 12
   - 12 +

2. *Please tick the colour that you like most from this list:*
   - Blue
   - Green
   - Yellow
   - Orange
   - Black

**Solution**

- The questionnaire is easy to fill in and the data will be easy to collect and analyse.

- An adult could answer question 1 and you would not be aware of this, as the 12 + category would include children and adults.

- The age categories overlap. A 12-year old would not know whether to tick the second or third box.

- The survey only asks for the preferred colour from a limited choice. If you want to find the *favourite* colour, you will have to give many more choices (red and purple, for example, are colours which people might want to choose as their favourites).

- An alternative way of improving the second question would be to have an extra category labelled

   "*Other colour (please specify) "
Example 2

Comment on the questions below:

1. Do cars cause pollution in the city centre? YES ☐ NO ☐

2. Do cars cause traffic hold-ups in the city centre? YES ☐ NO ☐

3. Are some car drivers a danger to pedestrians in the city centre? YES ☐ NO ☐

4. Do you think that cars should be banned from the city centre? YES ☐ NO ☐

Solution

The questions are biased. The first three are designed to focus on the disadvantages and dangers of cars, so that people are more likely to say 'yes' when they answer question 4.

Exercises

1. Design a questionnaire to find out whether people would be in favour of banning cars from your nearest city centre.

2. Design a questionnaire that could be used to investigate students' opinions of the method of transport that they use to travel to school.

3. Design a questionnaire to investigate how students rate the quality of the meals served in your school canteen.

4. Design a questionnaire that can be used to determine whether the general public, in your area, would be in favour of building a new youth club.

5. Rewrite the following questions so that they are not biased in any way:
   (a) Do you agree that maths is boring?
   (b) Are you in favour of town centre car park charges being increased in order to discourage car drivers from using their cars?
   (c) The price of a school lunch has not increased for 2 years. Do you think that school lunches are good value for money?
6. Comment critically on the following questions. In each case, rewrite the question to show the improvements you have made.

(a) *Are you young, middle-aged or old?*

(b) *Please select your favourite breakfast cereal from this list:*

- Cornflakes □
- Rice Crispies □
- Frosties □
- Bran Flakes □

(c) *How old are you?*

- 0 → 5 □
- 7 → 10 □
- 12 + □

(d) *Do you have any brothers?*

- Do you have one brother?
- Do you have more than one brother?
- Do you have at least two brothers?

7. Design biased questionnaires that would encourage people to reach the conclusion that the government:

(a) dislikes motorists,

(b) encourages motorists.

8. The local council interviews people who are arriving in the city centre on a warm, sunny day. People are asked which of the following methods of transport they have used:

- Walking □
- Cycling □
- Bus □
- Car □

(a) Explain why the results may not be reliable for deciding transport policies.

(b) Suggest how the council should collect more data.

9. As part of a survey, children were asked on which days they watched television during the previous week.

(a) Describe what problems the researchers may have had in reporting on how much television these children watched.

(b) Design a better question or questions to gather data for a report on how much television children watch.

10. Design a questionnaire that could be used to gather data on how school children spend their summer holidays.
20.2 Data Display

In this section we revise how to display data visually using bar charts, vertical line diagrams, pie charts and pictograms.

Example 1

At the end of each half term, the students in a particular school are graded A, B or C for their reports. The table shows the grades awarded for one class.

Illustrate this data using a pie chart.

Solution

The angles must be calculated first:

<table>
<thead>
<tr>
<th>Grade</th>
<th>Frequency</th>
<th>Angle</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>12</td>
<td>$\frac{12}{30} \times 360^\circ = 144^\circ$</td>
</tr>
<tr>
<td>B</td>
<td>16</td>
<td>$\frac{16}{30} \times 360^\circ = 192^\circ$</td>
</tr>
<tr>
<td>C</td>
<td>2</td>
<td>$\frac{2}{30} \times 360^\circ = 24^\circ$</td>
</tr>
<tr>
<td>TOTALS</td>
<td>30</td>
<td>360°</td>
</tr>
</tbody>
</table>

The pie chart can then be drawn as shown below:
Both the table and the pie chart show that very few of the pupils in this class were awarded a poor grade (C), and that B was the grade most frequently awarded to this class.

Example 2

A school canteen offers students a choice of burgers, sandwiches, baked potatoes or the meal of the day. On one day the number of children making each choice was recorded; the table below shows the children's choices for that day.

<table>
<thead>
<tr>
<th>Choice</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sandwiches</td>
<td>160</td>
</tr>
<tr>
<td>Burgers</td>
<td>45</td>
</tr>
<tr>
<td>Baked Potatoes</td>
<td>90</td>
</tr>
<tr>
<td>Meal of the Day</td>
<td>225</td>
</tr>
</tbody>
</table>

Illustrate this data using:

(a) a *pictogram*,

(b) a *bar chart*.

**Solution**

(a) *Pictogram to Show the Choices Made in a School Canteen*

<table>
<thead>
<tr>
<th>Choice</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sandwiches</td>
<td>160</td>
</tr>
<tr>
<td>Burgers</td>
<td>45</td>
</tr>
<tr>
<td>Baked Potatoes</td>
<td>90</td>
</tr>
<tr>
<td>Meal of the Day</td>
<td>225</td>
</tr>
</tbody>
</table>

Illustrate this data using:

(a) a *pictogram*,

(b) a *bar chart*.
Example 3

The pupils in class 7C were given a short maths test. The scores are listed in the following table:

<table>
<thead>
<tr>
<th>Score</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>Frequency</td>
<td>0</td>
<td>2</td>
<td>3</td>
<td>2</td>
<td>5</td>
<td>3</td>
<td>7</td>
<td>6</td>
<td>3</td>
<td>0</td>
<td>2</td>
</tr>
</tbody>
</table>

Illustrate these results on a vertical line diagram.
Solution

Vertical Line Diagram to Show the Results of a Maths Test for Pupils in Class 7C

Exercises

1. Ameer visits a car park and records the registration letter of each of the cars there. The results are given in the table below:

<table>
<thead>
<tr>
<th>Letter</th>
<th>P</th>
<th>R</th>
<th>S</th>
<th>T</th>
<th>V</th>
<th>W</th>
<th>OTHER</th>
</tr>
</thead>
<tbody>
<tr>
<td>Frequency</td>
<td>3</td>
<td>6</td>
<td>7</td>
<td>2</td>
<td>5</td>
<td>3</td>
<td>10</td>
</tr>
</tbody>
</table>

Illustrate Ameer’s data using a pie chart.

2. Pupils in class 8B record the number of bus journeys made by each member of their class during one week. The results are listed in the following table:

<table>
<thead>
<tr>
<th>No. of Journeys</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
</tr>
</thead>
<tbody>
<tr>
<td>Frequency</td>
<td>6</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>14</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

(a) Illustrate this data using a vertical line diagram.

(b) Give a possible reason why the category '10 journeys' has such a high frequency.
3. A survey was carried out to see how children in a school rated their school bus service. The results are listed in the following table:

<table>
<thead>
<tr>
<th>Response</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>Very good</td>
<td>1</td>
</tr>
<tr>
<td>Good</td>
<td>7</td>
</tr>
<tr>
<td>Satisfactory</td>
<td>26</td>
</tr>
<tr>
<td>Bad</td>
<td>24</td>
</tr>
<tr>
<td>Very bad</td>
<td>2</td>
</tr>
</tbody>
</table>

Illustrate these results with a pictogram. Write a brief note to the bus company commenting on what the survey shows about the quality of the service they provide to the school.

4. Mandy draws the following bar chart to illustrate the ages of some of her friends:

(a) How many of Mandy's friends are included on the bar chart?
(b) Mandy is 13 years old. How many of her friends are the same age as she is?
(c) How many of her friends are aged 14 or younger?
5. A class carried out a survey to find out how many TV sets there were in each of their homes. The results are shown in the following table:

<table>
<thead>
<tr>
<th>Number of TV Sets</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Frequency</td>
<td>1</td>
<td>8</td>
<td>14</td>
<td>2</td>
<td>3</td>
<td>1</td>
</tr>
</tbody>
</table>

Illustrate this data using a bar chart, and comment on the results of the survey.

6. Jason has £3 per week pocket money. One week, Jason spent his £3 pocket money on the following items:

- Sweets: 60p
- Football club: £1.00
- Tennis club: 50p
- Comics: 70p
- New pencil: 20p

(a) Explain why a pie chart would be a good way to show how Jason spent his pocket money that week.
(b) Draw a pie chart to show how Jason spent that week's pocket money.

7. The pupils in a class carry out a survey to determine their favourite types of television programme. The results are given in the following table:

- Soaps: 8
- Films: 9
- News programmes: 1
- Quizzes: 2
- Wildlife programmes: 4
- Others: 6

(a) Illustrate this data using a suitable diagram.
(b) Explain why you chose to illustrate the data using this method.
8. A pocket money survey for a group of 50 students produced the following results:

<table>
<thead>
<tr>
<th>Weekly Pocket Money</th>
<th>£1</th>
<th>£2</th>
<th>£3</th>
<th>£5</th>
<th>£10</th>
</tr>
</thead>
<tbody>
<tr>
<td>Frequency</td>
<td>10</td>
<td>22</td>
<td>12</td>
<td>5</td>
<td>1</td>
</tr>
</tbody>
</table>

(a) Illustrate this data using a suitable diagram.
(b) Explain why you chose this type of diagram to illustrate the data.
(c) Sam gets £4 pocket money each week. Would Sam be wise to use the results of this survey to support his request for an increase in his pocket money?

9. (a) Carry out a survey to find the favourite type of chocolate bar for your class.
(b) Illustrate your results with a suitable diagram.
(c) Comment on your results.

10. (a) Collect data on the age, in years and months, of each member of your class.
(b) Illustrate the data with a suitable diagram.
(c) Comment on the results you obtain.

20.3 Line Graphs

In this section we look at how to use line graphs.

Example 1

Mr Smith recorded the temperature outside his classroom every hour during one school day. His results are listed in the following table:

<table>
<thead>
<tr>
<th>Time</th>
<th>0900</th>
<th>1000</th>
<th>1100</th>
<th>1200</th>
<th>1300</th>
<th>1400</th>
<th>1500</th>
<th>1600</th>
</tr>
</thead>
<tbody>
<tr>
<td>Temperature (° C)</td>
<td>8</td>
<td>9</td>
<td>11</td>
<td>12</td>
<td>12</td>
<td>13</td>
<td>10</td>
<td>9</td>
</tr>
</tbody>
</table>

(a) Plot this data using a line graph.
(b) Estimate the temperature at 1030 and at 1130.
Solution

(a) First plot the points that represent each of the recorded temperatures and then join these points with straight lines, as shown in the following diagram:

(b) The broken lines added to the graph show how to estimate the temperature at other times. The estimate for the temperature at

1030 is 10 ° and at 1130 is 11.5 °

Example 2

The following line graph shows how much rain had fallen by certain times one day:
Estimate the amount of rain that had fallen by:
(a) 1030  
(b) 1300
(c) 1900  
(d) between 1300 and 1900

During which times did no rain fall?

**Solution**

The broken lines on the following graph show how to obtain the estimates:

Estimates:  
(a) By 1030, 10 mm of rain had fallen.
(b) By 1300, 17.5 mm of rain had fallen.
(c) By 1900, 32.5 mm of rain had fallen.
(d) This means that approximately $32.5 - 17.5 = 15$ mm of rain fell between 1300 and 1900.

The horizontal sections of the graph indicate that it did not rain at all between 1000 and 1100 and also between 1400 and 1500. (There may have been other short periods of time when it did not rain.)
Exercises

1. The outside temperature was recorded every hour during a school day. The results are given below:

<table>
<thead>
<tr>
<th>Time</th>
<th>0900</th>
<th>1000</th>
<th>1100</th>
<th>1200</th>
<th>1300</th>
<th>1400</th>
<th>1500</th>
<th>1600</th>
</tr>
</thead>
<tbody>
<tr>
<td>Temperature (° C)</td>
<td>12</td>
<td>13</td>
<td>16</td>
<td>18</td>
<td>22</td>
<td>21</td>
<td>20</td>
<td>18</td>
</tr>
</tbody>
</table>

(a) Draw a line graph for this data.
(b) Estimate the temperature at 1230 and at 1530.

2. The height of a plant was measured regularly after it had been transplanted, and the results are given below:

<table>
<thead>
<tr>
<th>Day</th>
<th>4</th>
<th>8</th>
<th>12</th>
<th>16</th>
<th>20</th>
<th>24</th>
</tr>
</thead>
<tbody>
<tr>
<td>Height (cm)</td>
<td>2</td>
<td>3</td>
<td>6</td>
<td>10</td>
<td>12</td>
<td>14</td>
</tr>
</tbody>
</table>

(a) Draw a line graph to show how the height of the plant increased.
(b) Estimate the height of the plant after:
   (i) 10 days
   (ii) 18 days
   (iii) 13 days

3. A motorist on a long journey recorded the distances that he had travelled by various times:

<table>
<thead>
<tr>
<th>Time</th>
<th>0800</th>
<th>0900</th>
<th>1000</th>
<th>1100</th>
<th>1200</th>
<th>1400</th>
</tr>
</thead>
<tbody>
<tr>
<td>Distance Travelled (miles)</td>
<td>0</td>
<td>60</td>
<td>100</td>
<td>150</td>
<td>210</td>
<td>320</td>
</tr>
</tbody>
</table>

(a) Draw a line graph for this data.
(b) Estimate the distance travelled at:
   (i) 0830
   (ii) 1130
   (iii) 1300

4. Rachel keeps a record of the mass of her puppy as it grows. The records she has gathered are listed below:

<table>
<thead>
<tr>
<th>Age of Puppy (months)</th>
<th>1</th>
<th>2</th>
<th>4</th>
<th>6</th>
<th>9</th>
<th>10</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mass (kg)</td>
<td>5</td>
<td>8</td>
<td>11</td>
<td>12</td>
<td>20</td>
<td>23</td>
<td>24</td>
</tr>
</tbody>
</table>

Use a line graph to estimate the mass of the puppy when its age is:
(a) 3 months
(b) 8 months,
(c) 11 months.
5. During a flood alert the depth of water in a river was measured several times. The times and depths were recorded as shown below:

<table>
<thead>
<tr>
<th>Time</th>
<th>0700</th>
<th>0900</th>
<th>1200</th>
<th>1600</th>
<th>2000</th>
<th>2200</th>
<th>2400</th>
</tr>
</thead>
<tbody>
<tr>
<td>Depth (m)</td>
<td>1.4</td>
<td>1.8</td>
<td>1.9</td>
<td>2.2</td>
<td>2.6</td>
<td>3.0</td>
<td>2.8</td>
</tr>
</tbody>
</table>

Use a line graph to estimate the depth of the river at:
(a) 1000 (b) 1800 (c) 2300

6. The depth of water in a harbour was recorded at various times during one day. The data recorded is listed below:

<table>
<thead>
<tr>
<th>Time</th>
<th>0600</th>
<th>0900</th>
<th>1000</th>
<th>1200</th>
<th>1600</th>
<th>1800</th>
<th>2000</th>
<th>2400</th>
</tr>
</thead>
<tbody>
<tr>
<td>Depth (m)</td>
<td>3</td>
<td>1.8</td>
<td>1.2</td>
<td>0.8</td>
<td>2.1</td>
<td>3.2</td>
<td>2.4</td>
<td>1.9</td>
</tr>
</tbody>
</table>

Use a line graph to estimate the depth of water in the harbour at:
(a) 0800 (b) 1400 (c) 2200

7. The following table shows records of a patient's temperature while she was in hospital:

<table>
<thead>
<tr>
<th>Day</th>
<th>Wednesday</th>
<th>Thursday</th>
<th>Friday</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time</td>
<td>0600</td>
<td>1200</td>
<td>1800</td>
</tr>
<tr>
<td>Temperature (°C)</td>
<td>38.7</td>
<td>39.1</td>
<td>39.4</td>
</tr>
</tbody>
</table>

Use a line graph to estimate when the patient's temperature:
(a) rose above 39.5 °C,
(b) fell below 38 °C.

8. In a science experiment, masses are hung on the end of a spring and the length of the spring is measured. The results are recorded in the following table:

<table>
<thead>
<tr>
<th>Mass (gms)</th>
<th>50</th>
<th>100</th>
<th>200</th>
<th>400</th>
</tr>
</thead>
<tbody>
<tr>
<td>Length (cm)</td>
<td>5.6</td>
<td>6.3</td>
<td>7.7</td>
<td>10.5</td>
</tr>
</tbody>
</table>

Use a line graph to estimate the length of the spring for:
(a) a 150 gram mass (b) a 300 gram mass (c) a 500 gram mass.
9. The table opposite lists the increase in the average temperature of the earth since 1800.

<table>
<thead>
<tr>
<th>Year</th>
<th>Temperature Increase (°C)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1860</td>
<td>0.03</td>
</tr>
<tr>
<td>1920</td>
<td>0.06</td>
</tr>
<tr>
<td>1940</td>
<td>0.10</td>
</tr>
<tr>
<td>1960</td>
<td>0.18</td>
</tr>
<tr>
<td>1980</td>
<td>0.32</td>
</tr>
<tr>
<td>2000</td>
<td>0.60</td>
</tr>
</tbody>
</table>

Use a line graph to estimate the temperature increase by the year:
(a) 1950 (b) 1990 (c) 2020

10. Julie goes on a 5 week diet. She records her mass every 5 days.

<table>
<thead>
<tr>
<th>Day</th>
<th>0</th>
<th>5</th>
<th>10</th>
<th>15</th>
<th>20</th>
<th>25</th>
<th>30</th>
<th>35</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mass (kg)</td>
<td>74</td>
<td>73</td>
<td>71</td>
<td>68</td>
<td>68</td>
<td>68</td>
<td>67</td>
<td>64</td>
</tr>
</tbody>
</table>

Use a line graph to determine when Julie's mass dropped to:
(a) 72 kg (b) 70 kg (c) 65 kg