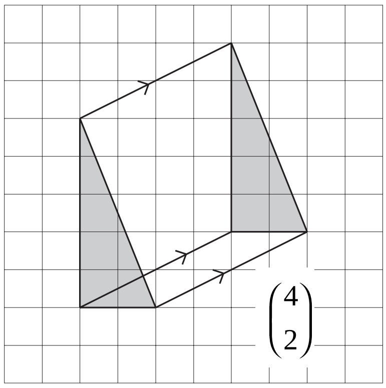
19 Vectors

19.1 Vectors and Scalars

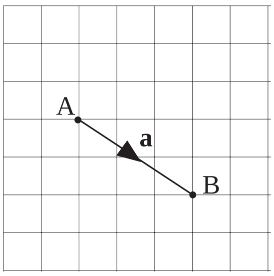
Vectors were used in an earlier chapter to describe translations. The diagram shows the translation of a triangle by the vector $\begin{pmatrix} 4 \\ 2 \end{pmatrix}$.



Note that the vector $\begin{pmatrix} 4 \\ 2 \end{pmatrix}$ specifies *how far* the triangle is to be moved and the *direction*, i.e. 4 units *along* and 2 units *up*.

All vectors have length or size and direction. Quantities which do not have direction, but only length or size are known as *scalar quantities*. Quantities like *mass*, *length*, *area* and *speed* are *scalars* because they have size only, while quantities like *force* and *velocity* are *vectors* because they have a direction as well as a size.

The two points A and D are shown in the diagram. The displacement of B from A is a vector because it has length and a direction.



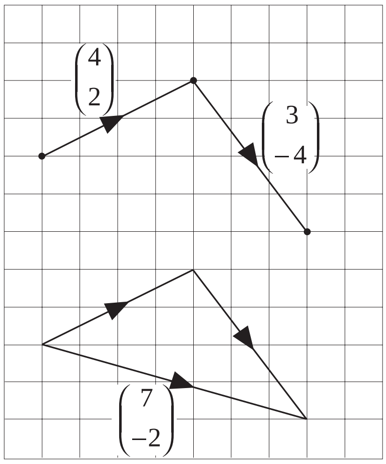
We can write this displacement as $\vec{AB} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$ or label the vector **a** and write

$\mathbf{a} = \vec{AB}$ or $\mathbf{a} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$.

The notation **a** is used when **a** is a vector and the notation *a* is used when *a* is a scalar.

Vectors can simply be added and subtracted.

Consider $\begin{pmatrix}4\\2\end{pmatrix}+\begin{pmatrix}3\\-4\end{pmatrix}$ which can be represented as shown in the following diagram.



So, from the diagram, the *addition* of these two vectors can be written as a single vector $\begin{pmatrix}7\\-2\end{pmatrix}$, which is just the addition of each component of the original vector. In general,

$$\begin{pmatrix}a\\b\end{pmatrix}+\begin{pmatrix}c\\d\end{pmatrix}=\begin{pmatrix}a+c\\b+d\end{pmatrix}$$

A similar result is true for *subtraction*,

$$\begin{pmatrix}a\\b\end{pmatrix}-\begin{pmatrix}c\\d\end{pmatrix}=\begin{pmatrix}a-c\\b-d\end{pmatrix}$$

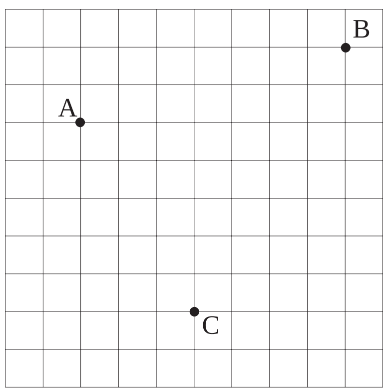
A vector can be *multiplied* by a scalar, i.e. a number, by multiplying each component by that scalar.

For example, $4\times\begin{pmatrix}3\\-2\end{pmatrix}=\begin{pmatrix}12\\-8\end{pmatrix}$.

In general,

$$k\times\begin{pmatrix}a\\b\end{pmatrix}=\begin{pmatrix}ka\\kb\end{pmatrix}$$

Worked Example 1



Write each of the following vectors in the form $\begin{pmatrix}a\\b\end{pmatrix}$.

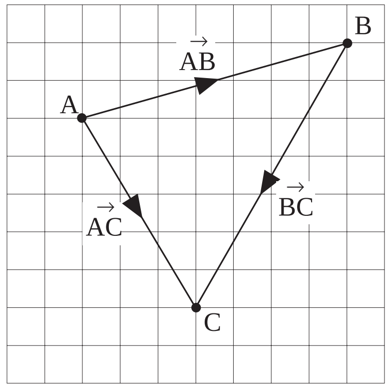
(a) $\overrightarrow{AB}$

(b) $\overrightarrow{BC}$

(c) $\overrightarrow{AC}$

Solution

From the diagram:



(a) $\overrightarrow{AB} = \begin{pmatrix} 7 \\ 2 \end{pmatrix}$

(b) $\overrightarrow{BC} = \begin{pmatrix} -4 \\ -7 \end{pmatrix}$

(c) $\overrightarrow{AC} = \begin{pmatrix} 3 \\ -5 \end{pmatrix}$

Worked Example 2

If $\mathbf{a} = \begin{pmatrix} 3 \\ 7 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 2 \\ -4 \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} -3 \\ -4 \end{pmatrix}$ find:

(a) $2\mathbf{a}$ (b) $\mathbf{b} + \mathbf{c}$ (c) $\mathbf{a} - \mathbf{b}$ (d) $2\mathbf{a} + 3\mathbf{b}$

Solution

(a) $2\mathbf{a} = 2 \times \begin{pmatrix} 3 \\ 7 \end{pmatrix} = \begin{pmatrix} 2 \times 3 \\ 2 \times 7 \end{pmatrix} = \begin{pmatrix} 6 \\ 14 \end{pmatrix}$

(b) $\mathbf{b} + \mathbf{c} = \begin{pmatrix} 2 \\ -4 \end{pmatrix} + \begin{pmatrix} -3 \\ -4 \end{pmatrix} = \begin{pmatrix} 2 + (-3) \\ -4 + (-4) \end{pmatrix} = \begin{pmatrix} -1 \\ -8 \end{pmatrix}$

(c) $\mathbf{a} - \mathbf{b} = \begin{pmatrix} 3 \\ 7 \end{pmatrix} - \begin{pmatrix} 2 \\ -4 \end{pmatrix} = \begin{pmatrix} 3 - 2 \\ 7 - (-4) \end{pmatrix} = \begin{pmatrix} 1 \\ 11 \end{pmatrix}$

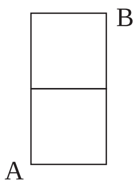
(d) $2\mathbf{a} + 3\mathbf{b} = 2 \times \begin{pmatrix} 3 \\ 7 \end{pmatrix} + 3 \times \begin{pmatrix} 2 \\ -4 \end{pmatrix} = \begin{pmatrix} 2 \times 3 \\ 2 \times 7 \end{pmatrix} + \begin{pmatrix} 3 \times 2 \\ 3 \times (-4) \end{pmatrix} = \begin{pmatrix} 6 \\ 14 \end{pmatrix} + \begin{pmatrix} 6 \\ -12 \end{pmatrix}$

$$= \begin{pmatrix} 6 + 6 \\ 14 + (-12) \end{pmatrix} = \begin{pmatrix} 12 \\ 2 \end{pmatrix}$$

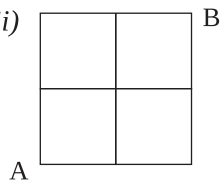
Just for Fun

Consider the following cases.

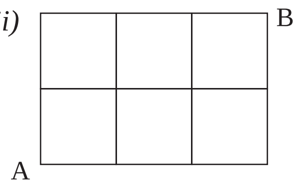
(i)



(ii)



(iii)



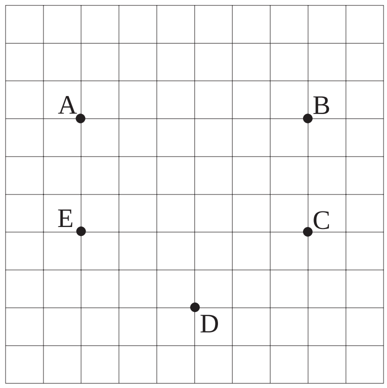
In each case, find the number of distinct ways to move from A *to* B.

Exercises

1. Which of the following are vectors and which are scalars:

(a)	Time	(b)	Velocity	(c)	Speed
(d)	Force	(e)	Distance	(f)	Temperature

2. Use the points in the grid below to write the vectors given in column vector form.



(a)	$\overrightarrow{AB}$	(b)	$\overrightarrow{AC}$
(c)	$\overrightarrow{DE}$	(d)	$\overrightarrow{BE}$
(e)	$\overrightarrow{EB}$	(f)	$\overrightarrow{AD}$
(g)	$\overrightarrow{CD}$	(h)	$\overrightarrow{DC}$

What is the relationship between

$\overrightarrow{AC}$ and $\overrightarrow{CA}$?

3. Plot the positions of the points A, B, C, D, E and F relative to a point O if:

(a)	$\overrightarrow{OA} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$	(b)	$\overrightarrow{OB} = \begin{pmatrix} 5 \\ -2 \end{pmatrix}$	(c)	$\overrightarrow{AC} = \begin{pmatrix} -3 \\ 2 \end{pmatrix}$
(d)	$\overrightarrow{BD} = \begin{pmatrix} -2 \\ -2 \end{pmatrix}$	(e)	$\overrightarrow{CE} = \begin{pmatrix} -4 \\ -1 \end{pmatrix}$	(f)	$\overrightarrow{DF} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$

Write the vector $\overrightarrow{EF}$ as a column vector.

4. If $\mathbf{a} = \begin{pmatrix} 4 \\ 7 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 3 \\ -5 \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} 0 \\ 4 \end{pmatrix}$, find:

(a)	$\mathbf{a} + \mathbf{b}$	(b)	$\mathbf{b} + \mathbf{c}$	(c)	$\mathbf{a} + \mathbf{c}$
(d)	$\mathbf{a} - \mathbf{b}$	(e)	$\mathbf{b} - \mathbf{a}$	(f)	$\mathbf{c} - \mathbf{a}$
(g)	$3\mathbf{a}$	(h)	$-2\mathbf{b}$	(i)	$4\mathbf{c}$
(j)	$2\mathbf{a} + 3\mathbf{b}$	(k)	$5\mathbf{c} - 3\mathbf{a}$	(l)	$4\mathbf{b} - 2\mathbf{c}$

5. If $\mathbf{a} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} -2 \\ -4 \end{pmatrix}$, find:

(a)	$3\mathbf{a} + 2\mathbf{b}$	(b)	$4\mathbf{a} + 3\mathbf{c}$	(c)	$6\mathbf{a} - 3\mathbf{b}$
(d)	$4\mathbf{c} + 2\mathbf{a}$	(e)	$3\mathbf{c} - 2\mathbf{a}$	(f)	$3\mathbf{a} + 2\mathbf{b} - 5\mathbf{c}$

6. If $\mathbf{a} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} 4 \\ -3 \end{pmatrix}$, solve the equations below to find the column vector **x**.

(a) $\mathbf{a} + \mathbf{x} = \mathbf{b}$ (b) $\mathbf{x} - \mathbf{c} = \mathbf{a}$ (c) $\mathbf{x} + \mathbf{b} = \mathbf{c}$

(d) $2\mathbf{x} + \mathbf{a} = \mathbf{b}$ (e) $3\mathbf{a} + 2\mathbf{x} = \mathbf{c}$ (f) $4\mathbf{a} - \mathbf{x} = \mathbf{c}$

(g) $3\mathbf{a} + 2\mathbf{x} = 4\mathbf{b}$ (h) $\mathbf{a} - 2\mathbf{x} = 4\mathbf{c}$ (i) $3\mathbf{b} + 2\mathbf{x} = \mathbf{c}$

7. In this question $\mathbf{a} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$.

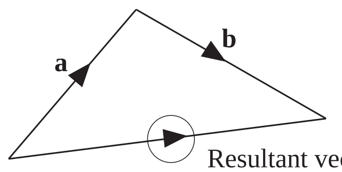
For each part draw the vectors listed on separate diagrams.

(a) $\mathbf{a}$, $\mathbf{b}$, $\mathbf{a} + \mathbf{b}$ (b) $\mathbf{a}$, $\mathbf{c}$, $\mathbf{a} - \mathbf{c}$ (c) $\mathbf{b}$, $2\mathbf{b}$, $3\mathbf{b}$

(d) $\mathbf{c}$, $-\mathbf{c}$, $-2\mathbf{c}$ (e) $\mathbf{a}$, $\mathbf{b}$, $2\mathbf{a} - 3\mathbf{b}$

19.2 Applications of Vectors

There are many applications of vectors, and in this section they are applied to *velocities* and *forces*. These quantities can be represented by vectors which can be added to find the resultant of the vectors, as shown below.



Resultant vector = $\mathbf{a} + \mathbf{b}$

Worked Example 1

The water in a river flows at a speed of 2 m/s. Tracey swims at 1 m/s at right angles to the bank. If the river is 4 m wide, find:

(a) the time it takes Tracey to cross the river,

(b) the speed she moves at relative to the bank,

(c) the distance she travels.

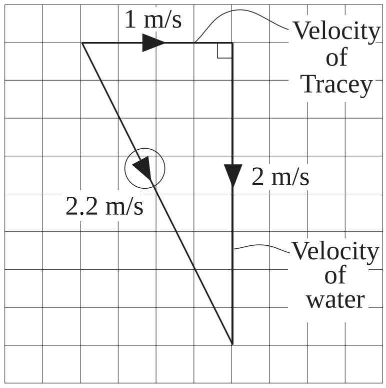
Solution

(a) Tracey travels 1 m across the river every second, so she takes 4 seconds to cross the river.

(b) The diagram shows the two vectors, for Tracey and for the water in the river. These have been drawn to scale and the resultant has length 2.2 m/s.

Tracey travels at a speed of 2.2 m/s and moves downstream as well as across the river.

(c) Tracey takes 4 seconds to cross the river. As her speed is 2.2 m/s, she travels a total of $4 \times 2.2 = 8.8$ m.



Worked Example 2

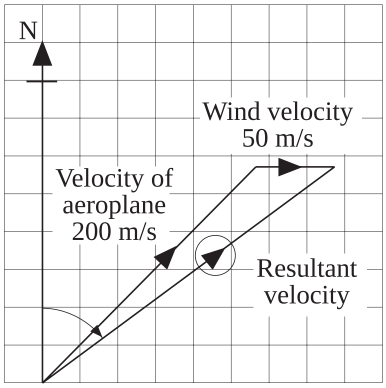
(a) An aeroplane travels NE at a speed of 200 m/s in still air. The aeroplane flies in the same way when a wind is causing the air to travel E at 50 m/s. Find the actual speed of the aeroplane.

(b) If the wind is still in the same direction with speed 50 m/s, and the pilot wishes to actually travel at 200 m/s in a NE direction, what speed and direction should the pilot be flying his plane?

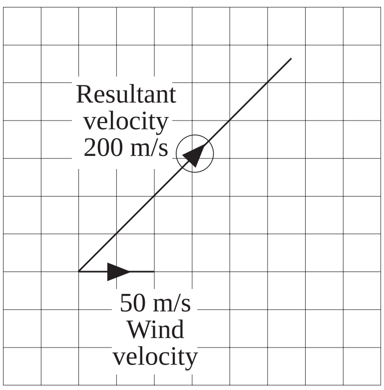
Solution

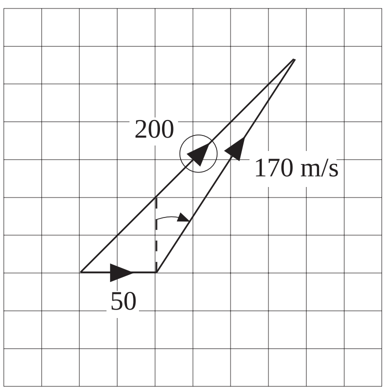
(a) The diagram shows two vectors representing the velocity of the aeroplane and the velocity of the wind.

The resultant velocity is a vector with estimated size 240 m/s and on an estimated bearing of $054°$.



(b) The diagram on the left shows the required resultant velocity of 200 m/s NE and wind velocity of 50 m/s E. The diagram on the right shows the velocity that the aeroplane must have to produce the required resultant velocity, which is estimated as 170 m/s on an estimated bearing of $033°$.





The last two examples have made use of scale drawing, but trigonometry or Pythagoras' Theorem can be used to solve problems once a sketch of the vectors has been drawn.

Worked Example 3

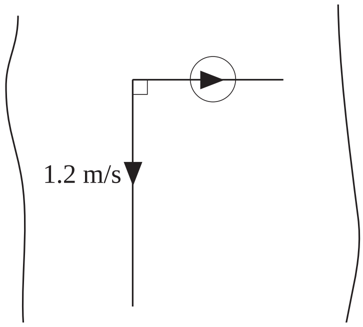
A man rows at 2 m/s across a river of width 20 m. The river flows at 1.2 m/s.

(a) Find the direction in which the man must row if he is to land on the other side, directly opposite his starting point.

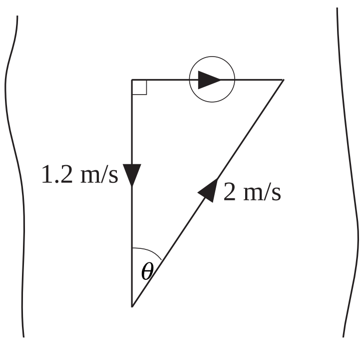
(b) Find the time that it takes for the man to row across the river.

Solution

(a) The first diagram shows the velocity of the water at 1.2 m/s downstream and the required direction of the resultant velocity.



The second diagram has the velocity of the boat, which is 2 m/s in an unknown direction. The problem is to find the angle θ.



But $$\cos\theta = \frac{1.2}{2.0}$$

$$= 0.6$$

giving $$\theta = 53.1°$$

So the man must row upstream at an angle of $53.1°$ to the bank.

(b) The resultant velocity is v m/s perpendicular to the bank as shown in the last diagram. In order to find the time to cross the river v must first be found.

Using Pythagoras,

$$2^2 = 1.2^2 + v^2$$

$$v^2 = 4 - 1.44$$

$$= 2.56$$

$$v = 1.6 \text{ m/s}$$

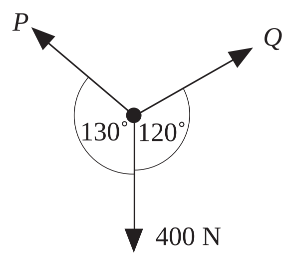
As the width of the river is 20 m the time can now be calculated as:

$$t = \frac{20}{1.6}$$
$$= 12.5 \text{ seconds.}$$

The next example shows how vectors can be applied to forces.

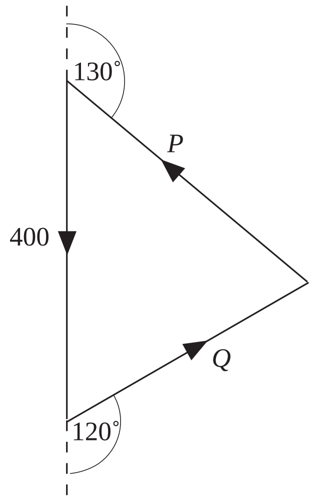
Worked Example 4

An object is supported by two cables. These exert forces of magnitude P and Q as shown in the diagram. Gravity exerts a downward force of 400 N on the object. Find P and Q if the resultant force is zero.



Solution

As the resultant force is zero the three force vectors should form a triangle as shown in the diagram.



The angles inside the triangle can be calculated as:

$60°, \; 50°$ and $70°$

Now using the sine rule gives:

$$\frac{400}{\sin 70°} = \frac{P}{\sin 60°} = \frac{Q}{\sin 50°}$$

so that

$$\frac{P}{\sin 60°} = \frac{400}{\sin 70°} \quad \text{and} \quad \frac{Q}{\sin 50°} = \frac{400}{\sin 70°}$$

$$P = \frac{400 \sin 60°}{\sin 70°} \qquad Q = \frac{400 \sin 50°}{\sin 70°}$$

$$= 369 \text{ N} \qquad = 326 \text{ N}$$

Similar results could have been obtained by using a scale drawing.

Exercises

1. A girl paddles her canoe so that it has a velocity of 3 m/s at right angles to the bank of a river. The water in the river moves at 1.5 m/s.

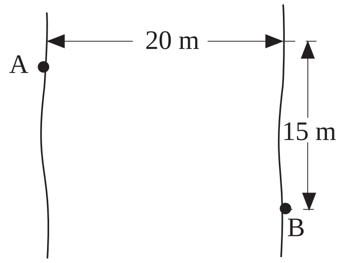
   (a) Use a scale drawing to find the speed at which the canoe actually travels and the angle its path makes with the bank.

   (b) How long does it take for the girl to cross the river, if its width is 10 m?

   (c) How far downstream does the girl travel?

2. Catherine paddles a canoe across a river. She starts at A, but travels downstream 15 m to reach the other side at B. It takes her 8 seconds to cross the river.

   If Catherine paddles at right angles to the bank, find her speed and the speed of the water.



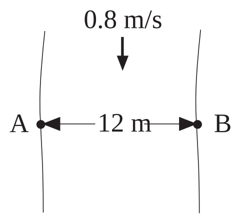
3. An aeroplane flies on a bearing of $060°$ and a speed of 50 m/s. It is blown off course by a wind of speed 20 m/s that blows due west.

   (a) Use a scale drawing to find the resultant velocity of the aeroplane.

   (b) Find the direction in which the aeroplane should fly, and at what speed, if it is to actually travel at 50 m/s on a bearing of $060°$.

4. Mya is going to swim across a river of width 12 m, in which the current is flowing at 0.8 m/s. She swims at a speed of 1.0 m/s. She wants to swim directly from A to B. In what direction should she swim and how long does it take her to cross the river?

   Obtain your answers to the question by scale drawing and calculation.



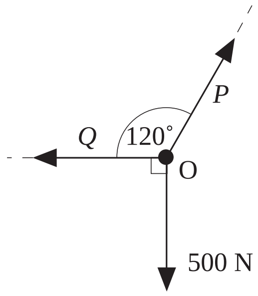
5. An aeroplane heads due north at 200 km/h, but is blown off course by a wind of 60 km/h from the east.

   (a) Find the direction in which the aeroplane actually travels and the resultant speed.

   (b) In what direction should the aeroplane fly if it is to travel due north?

6. A river estuary is 2 km wide and the water in it flows at 1 m/s when the tide is going out. A boat that can travel at 5 m/s in still water is to cross the river to a point exactly opposite on the other bank. Find the time it takes for the boat to cross the river estuary.

7. The object O is held in position by 2 cables that exert forces of magnitude P and Q. Gravity exerts a downward force of 500 N. The object remains at rest so that the resultant force on it is zero.

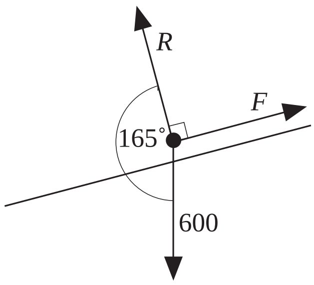


(a) Find P and Q by using a scale drawing.

(b) Calculate P and Q using trigonometry.

(c) Comment on the accuracy of your scale drawing.

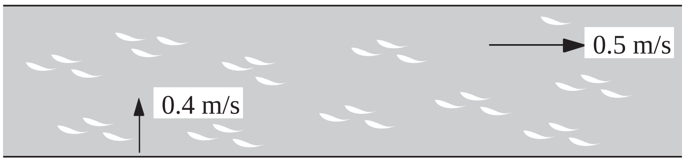
8. The diagram shows the forces acting on a skier travelling down a slope. If the resultant of the three forces is zero, find R and F.



9. Sarah can swim at 0.4 m/s in still water.

She tries to swim across a stream at right angles to the current.

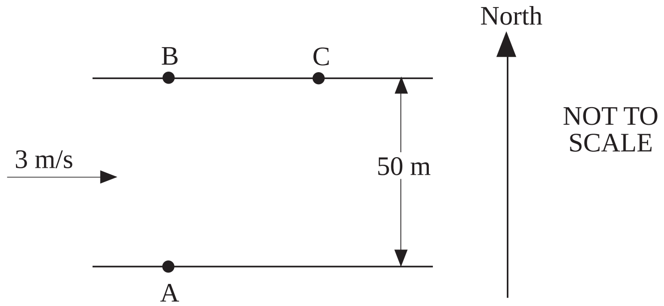
The speed of the current is 0.5 m/s.



Find Sarah's actual swimming speed in the stream.

(SEG)

10.



A river with parallel banks 50 m apart flows at a speed of 3 m/s from West to East. A girl can swim in still water with a speed of 1.6 m/s. She starts from A and intends to swim to B, which is due North of A and on the opposite bank. She heads North all the time but lands at C, further down stream.

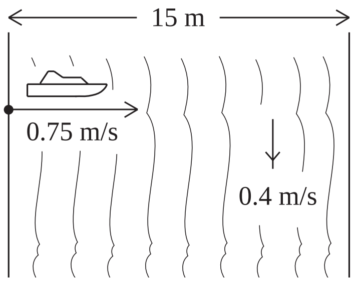
Calculate the distance BC.

(MEG)

11. A model power boat can travel at 0.75 m/s in still water.

It is released from a point P on the bank of a river which flows at 0.4 m/s. The river is 15 m wide.

The boat is aimed continually in a direction perpendicular to the flow of the river, as shown in the diagram.



(a) By scale drawing or by calculation, find

(i) the resultant speed of the boat;

(ii) the direction in which the boat actually travels across the river.

(b) (i) How far downstream from P does the boat land on the opposite bank?

(ii) How long does the boat take to cross the river?

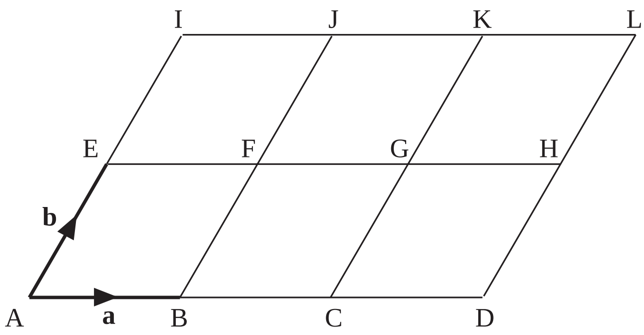
(NEAB)

Just for Fun

Peter begins at X and walks 1 step north, 2 steps east, 3 steps south, 4 steps west, 5 steps north, and continues in the same way, increasing the number of steps by one every time he turns right. Thus the rth leg of his journey is r steps long. If the number of legs of the journey he has travelled is n, then every time he turns south, what is his distance from X, in steps?

19.3 Vectors and Geometry

Vectors can be used to solve problems in geometry. In two dimensions, it is possible to describe the position of any point using two vectors. For example, using the vectors **a** and **b** shown in the diagram:



$\overrightarrow{AC} = 2\mathbf{a}$

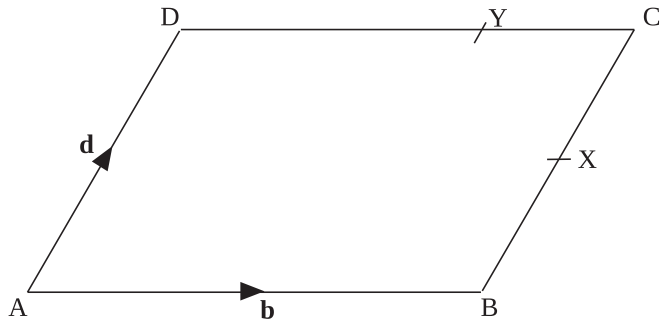
$\overrightarrow{AF} = \mathbf{a} + \mathbf{b}$

$\overrightarrow{AL} = 3\mathbf{a} + 2\mathbf{b}$

$\overrightarrow{LE} = -3\mathbf{a} - \mathbf{b}$

Worked Example 1

In the parallelogram shown below, $\overrightarrow{AB} = \mathbf{b}$ and $\overrightarrow{AD} = \mathbf{d}$. Also X is the midpoint of BC and Y lies on DC such that $DY = 2CY$.



Express the following vectors in terms of **b** and **d**.

(a) $\overrightarrow{AC}$ (b) $\overrightarrow{BX}$ (c) $\overrightarrow{AX}$

(d) $\overrightarrow{DY}$ (e) $\overrightarrow{AY}$ (f) $\overrightarrow{XY}$

Solution

(a) $\overrightarrow{AC} = \overrightarrow{AD} + \overrightarrow{BC}$

$= \mathbf{b} + \mathbf{d}$

(b) $\overrightarrow{BX} = \frac{1}{2}\overrightarrow{BC}$

$= \frac{1}{2}\mathbf{d}$

(c) $\overrightarrow{AX} = \overrightarrow{AB} + \overrightarrow{BX}$

$= \mathbf{b} + \frac{1}{2}\mathbf{d}$

(d) $\overrightarrow{DY} = \frac{2}{3}\overrightarrow{DC}$

$= \frac{2}{3}\mathbf{b}$

(e) $\overrightarrow{AY} = \overrightarrow{AD} + \overrightarrow{DY}$

$= \mathbf{d} + \frac{2}{3}\mathbf{b}$

(f) $\overrightarrow{XY} = \overrightarrow{XA} + \overrightarrow{AY}$

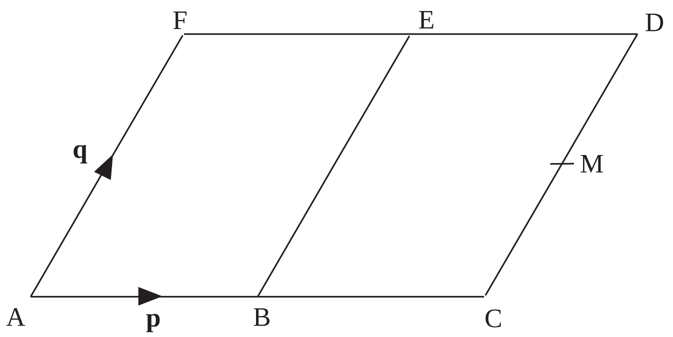
$$= -\left(\mathbf{b} + \frac{1}{2}\mathbf{d}\right) + \left(\mathbf{d} + \frac{2}{3}\mathbf{b}\right)$$

$$= \frac{2}{3}\mathbf{b} - \mathbf{b} + \mathbf{d} - \frac{1}{2}\mathbf{d}$$

$$= -\frac{1}{3}\mathbf{b} + \frac{1}{2}\mathbf{d}$$

Worked Example 2

The diagram shows 2 identical parallelograms. The vector $\mathbf{q} = \overrightarrow{AF}$ and the vector $\mathbf{p} = \overrightarrow{AB}$. The point M is the midpoint of CD.



(a) Show that BM is parallel to AD.

(b) Show that EM is parallel to FC.

Solution

(a) $\overrightarrow{BM} = \overrightarrow{BC} + \overrightarrow{CM}$

$$= \mathbf{p} + \frac{1}{2}\mathbf{q}$$

$\overrightarrow{AD} = \overrightarrow{AC} + \overrightarrow{CD}$

$$= 2\mathbf{p} + \mathbf{q}$$

As $\overrightarrow{AD} = 2\,\overrightarrow{BM}$, the lines AD and BM must be parallel.

(b) $\overrightarrow{EM} = \overrightarrow{ED} + \overrightarrow{DM}$

$$= \mathbf{p} + \left(-\frac{1}{2}\mathbf{q}\right)$$

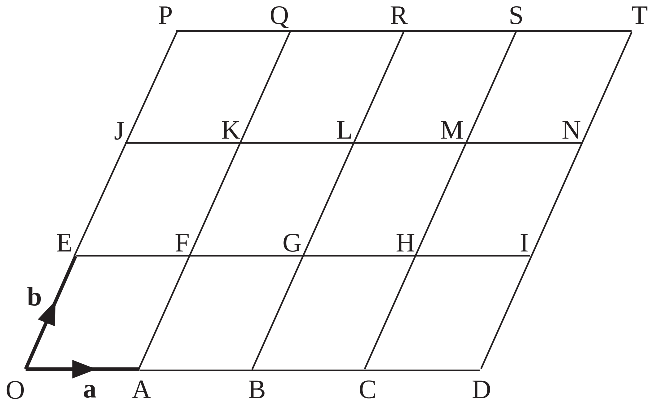
$$= \mathbf{p} - \frac{1}{2}\mathbf{q}$$

$$\overrightarrow{FC} = \overrightarrow{FD} + \overrightarrow{DC}$$
$$= 2\mathbf{p} - \mathbf{q}$$

As $\overrightarrow{FC} = 2\overrightarrow{EM}$, the lines FC and EM must be parallel.

Exercises

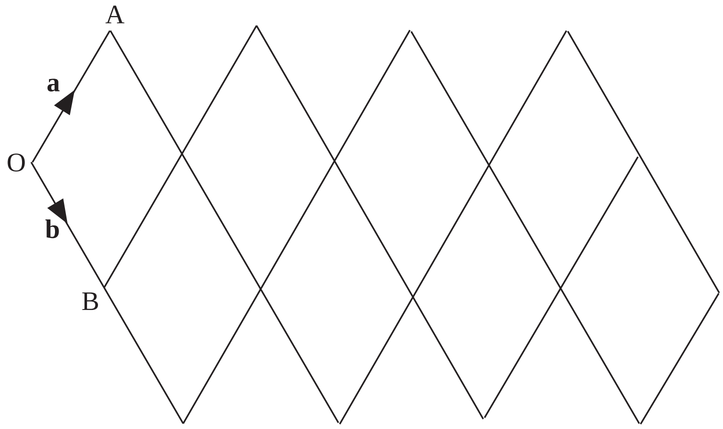
1. The diagram shows a grid made up of sets of equally spaced parallel lines. The vectors $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OE} = \mathbf{b}$ are shown on the grid.



Write each of the following vectors in terms of **a** and **b**.

(a)	$\overrightarrow{OD}$	(b)	$\overrightarrow{AB}$	(c)	$\overrightarrow{BG}$	(d)	$\overrightarrow{IS}$
(e)	$\overrightarrow{JP}$	(f)	$\overrightarrow{ES}$	(g)	$\overrightarrow{AQ}$	(h)	$\overrightarrow{CS}$
(i)	$\overrightarrow{PK}$	(j)	$\overrightarrow{PG}$	(k)	$\overrightarrow{RF}$	(l)	$\overrightarrow{SE}$
(m)	$\overrightarrow{CP}$	(n)	$\overrightarrow{GE}$	(o)	$\overrightarrow{IJ}$	(p)	$\overrightarrow{TA}$

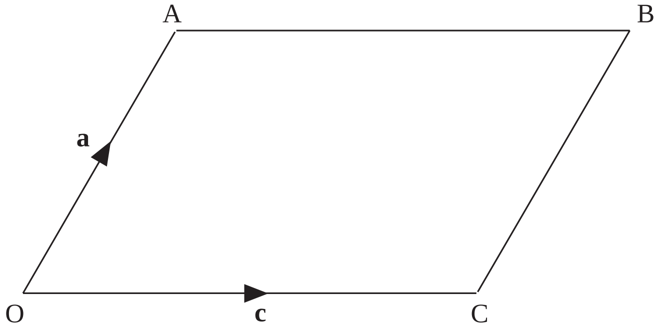
2. The diagram shows a grid made up of two sets of parallel lines. The vectors $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OE} = \mathbf{b}$ are shown on the grid.



Copy the grid and use the information below to label each point where lines meet.

$\overrightarrow{OC} = 2\mathbf{b}$	$\overrightarrow{OD} = \mathbf{a} + \mathbf{b}$	$\overrightarrow{DE} = \mathbf{a}$
$\overrightarrow{EF} = \mathbf{b}$	$\overrightarrow{BG} = \mathbf{a} + \mathbf{b}$	$\overrightarrow{BH} = 2\mathbf{a} + 3\mathbf{b}$
$\overrightarrow{HI} = -\mathbf{b}$	$\overrightarrow{IJ} = \mathbf{a} - \mathbf{b}$	$\overrightarrow{JK} = 3\mathbf{b} + \mathbf{a}$
$\overrightarrow{KL} = -\mathbf{a}$	$\overrightarrow{OM} = 4\mathbf{a} + 3\mathbf{b}$	$\overrightarrow{MN} = -\mathbf{a}$
$\overrightarrow{BP} = 3\mathbf{a} + 3\mathbf{b}$	$\overrightarrow{BQ} = \mathbf{a} + 2\mathbf{b}$	$\overrightarrow{DR} = 2\mathbf{a} + 4\mathbf{b}$

3. The diagram shows the parallelogram OABC, in which $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OC} = \mathbf{c}$.



(a) Write the following vectors in terms of **a** and **c**.

(i) $\overrightarrow{AB}$ (ii) $\overrightarrow{CB}$ (iii) $\overrightarrow{BC}$

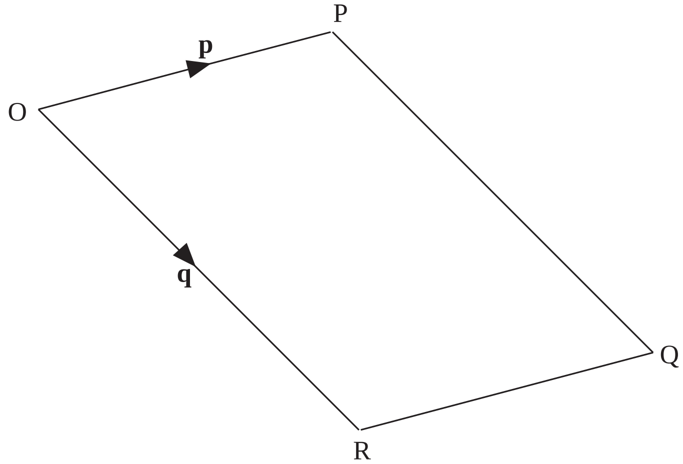
(iv) $\overrightarrow{AC}$ (v) $\overrightarrow{OB}$ (vi) $\overrightarrow{CA}$

(b) If X is the midpoint of AB and Y is the midpoint of BC, find the following in terms of **a** and **c**.

(i) $\overrightarrow{AX}$ (ii) $\overrightarrow{OX}$ (iii) $\overrightarrow{CY}$

(iv) $\overrightarrow{OY}$ (v) $\overrightarrow{XY}$

4. The diagram shows the parallelogram OPQR; the vectors $\mathbf{p} = \overrightarrow{OP}$ and $\mathbf{q} = \overrightarrow{OR}$.



(a) If M is the midpoint of PR, find $\overrightarrow{OM}$.

(b) If N is the midpoint of OQ, find $\overrightarrow{ON}$

(c) Comment on your answers to (a) and (b).

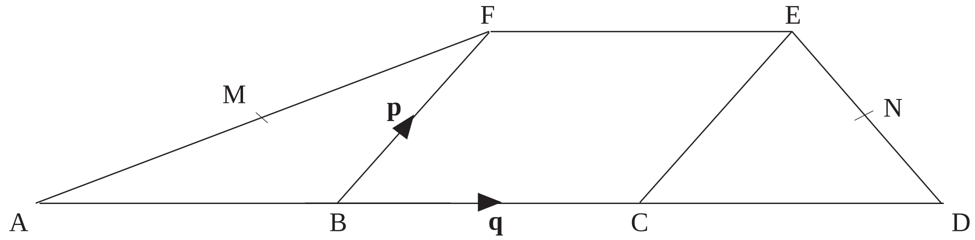
5. The diagram shows the rectangle OABC. The vectors **i** and **j** are such that $\overrightarrow{OA} = 6\mathbf{j}$ and $\overrightarrow{OC} = 8\mathbf{i}$.

(a) If the point D lies on AB such that $AD = 3DB$, find $\overrightarrow{AD}$ and $\overrightarrow{OD}$ in terms of **i** and **j**.

(b) If E lies on BC such that $2BE = EC$, find $\overrightarrow{CE}$ and $\overrightarrow{OE}$.

(c) The point M is the midpoint of DE. Find $\overrightarrow{OM}$.

6. In the diagram, BCEF is a parallelogram and $AB = BC = CD$.
The vector $\mathbf{p} = \overrightarrow{BF}$ and $\mathbf{q} = \overrightarrow{BC}$. The point M is the midpoint of AF and N is the midpoint of DE.



(a) Express $\overrightarrow{AM}$ and $\overrightarrow{AN}$ in terms of **p** and **q**.

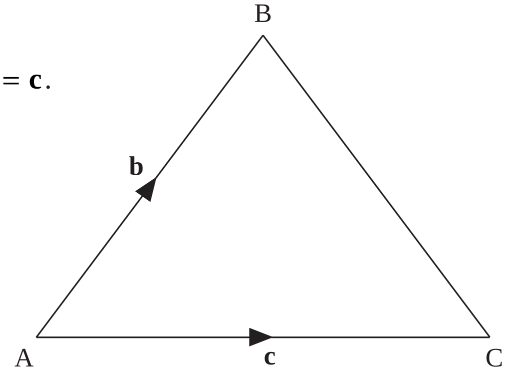
(b) Find $\overrightarrow{MN}$ in terms of **p** and **q** and explain why MN is parallel to AD.

7. In the triangle ABC, $\overrightarrow{AB} = \mathbf{b}$ and $\overrightarrow{AC} = \mathbf{c}$.

Use vectors to show that:

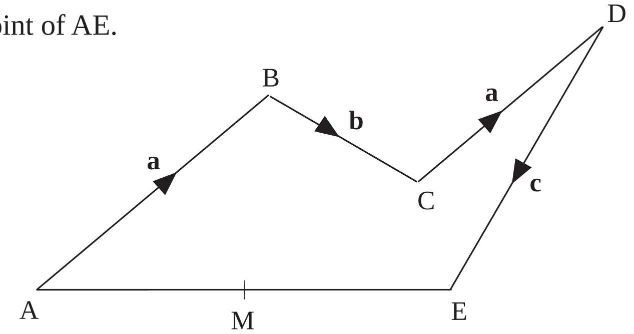
(a) a line joining the midpoint of AB and BC is parallel to AC,

(b) a line joining the midpoint of AB and AC is parallel to BC.



8. The shape in the diagram shows the vectors **a**, **b** and **c**.

M is the midpoint of AE.



Find each of the following in terms of **a**, **b** and **c**.

(a) $\overrightarrow{AE}$ (b) $\overrightarrow{AM}$ (c) $\overrightarrow{BA}$

(d) $\overrightarrow{MD}$ (e) $\overrightarrow{CM}$

9. ABCD is a parallelogram in which $\overrightarrow{AB} = \mathbf{b}$ and $\overrightarrow{AD} = \mathbf{d}$.

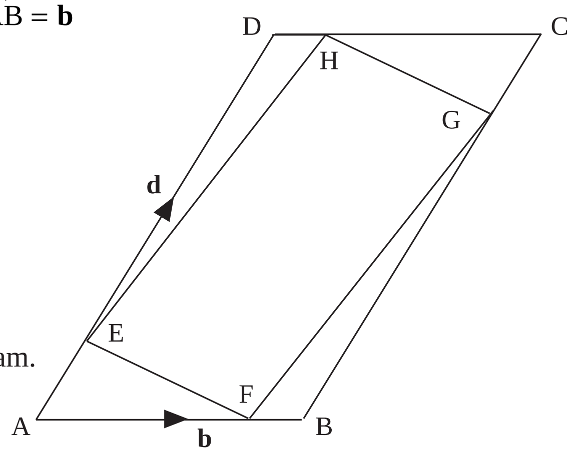
$AF = 4BF$

$BG = 4CG$

$CH = 4DH$

$DE = 4AE$

Show that EFGH is also a parallelogram.

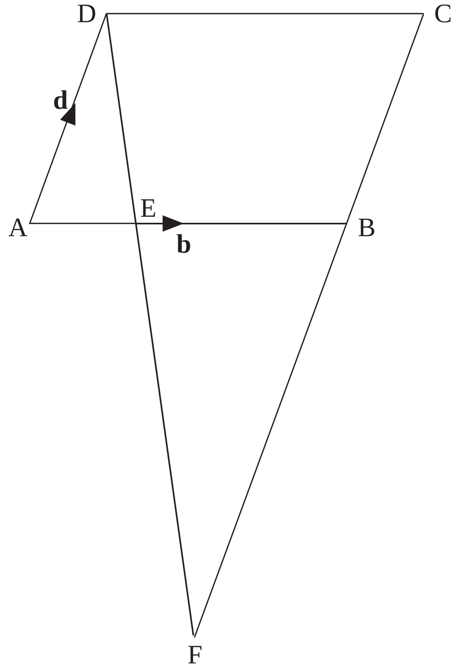


10. In the parallelogram ABCD, $\overrightarrow{AB} = \mathbf{b}$ and $\overrightarrow{AD} = \mathbf{d}$. Also $BE = 2AE$.

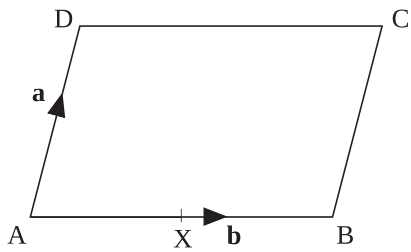
(a) Find $\overrightarrow{DE}$ and explain why $\overrightarrow{AF} = \mathbf{d} + \alpha\left(\frac{1}{3}\mathbf{b} - \mathbf{d}\right)$ for some values of α.

(b) Find $\overrightarrow{AC}$ and explain why $\overrightarrow{AF} = \mathbf{b} + \mathbf{d} - \beta\mathbf{d}$ for some value of β.

(c) Hence find the values of α and β.



11. In the parallelogram ABCD, $\overrightarrow{AD} = \mathbf{d}$ and $\overrightarrow{AB} = \mathbf{b}$. The point X is the midpoint of AB. The lines AC and DX intersect at Q.

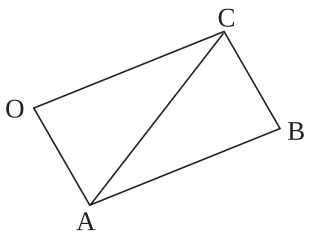


Find $\overrightarrow{AQ}$ in terms of **a** and **b**.

12. OABC is a parallelogram.

$\overrightarrow{OA} = 3\mathbf{p} - 2\mathbf{q}$

$\overrightarrow{OC} = 5\mathbf{p} + 6\mathbf{q}$



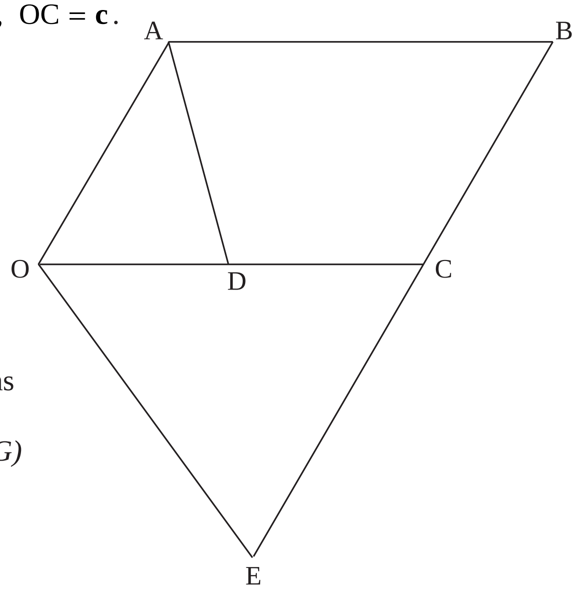
(a) Find $\overrightarrow{AC}$. Express your answer as simply as possible in terms of **p** and **q**.

(b) D is the point where $\overrightarrow{BD} = -2\mathbf{p} + 6\mathbf{q}$.

Using vector methods, show that D lies on the line AC produced.

(MEG)

13. OABC is a parallelogram. $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OC} = \mathbf{c}$.



BCE is a straight line, $\overrightarrow{BE} = 3\overrightarrow{BC}$.

D is the midpoint of OC.

(a) Write in terms of **a** and **c**,

(i) $\overrightarrow{AD}$,

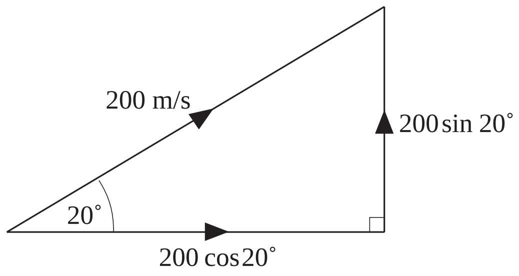
(ii) $\overrightarrow{OE}$.

(b) Deduce the ratio of the lengths of AD and OE.

(MEG)

19.4 Further Work with Vectors

In order to deal with vectors numerically, it is useful to be able to write vectors as *column vectors*. For example, consider the velocity of an aeroplane that travels at 200 m/s at an angle of $20°$ above the horizontal.



NOT TO SCALE

This velocity can be split into a horizontal component of $200\cos 20°$ and a vertical component of $200\sin 20°$. This can then be expressed as a column vector:

$$\begin{pmatrix} 200\cos 20° \\ 200\sin 20° \end{pmatrix} = \begin{pmatrix} 188 \\ 68 \end{pmatrix}$$

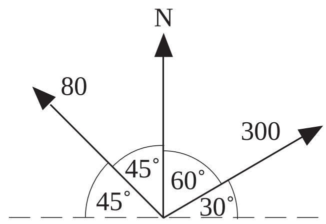
Rather than using drawings or applying trigonometry to diagrams, components can be used to obtain the answers to vector problems.

Worked Example 1

An aeroplane flies so that it would have a velocity of 300 m/s on a bearing of $060°$. A wind blows it off course. The velocity of the wind is 80 m/s NW. Find the resultant velocity of the aeroplane.

Solution

Using components that are east and north gives:



velocity of plane $= \begin{pmatrix} 300\cos 30° \\ 300\sin 30° \end{pmatrix}$

and

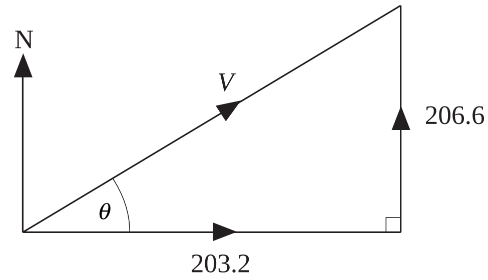
velocity of wind $= \begin{pmatrix} -80\cos 45° \\ 80\sin 45° \end{pmatrix}$

So the resultant velocity of the aeroplane is:

$$\begin{pmatrix} 300\cos 30° \\ 300\sin 30° \end{pmatrix} + \begin{pmatrix} -80\cos 45° \\ 80\cos 45° \end{pmatrix} = \begin{pmatrix} 300\cos 30° - 80\cos 45° \\ 300\sin 30° + 80\cos 45° \end{pmatrix}$$

$$= \begin{pmatrix} 203.2 \\ 206.6 \end{pmatrix}$$

The diagram shows the resultant velocity of the aeroplane.



By Pythagoras,

$$v^2 = 203.2^2 + 206.6^2$$

$$v = 290 \text{ m/s}$$

By trigonometry,

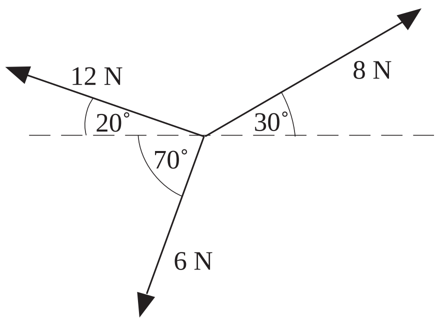
$$\tan\theta = \frac{206.6}{203.2}$$

$$\theta = 44.5°$$

So the resultant velocity is 290 m/s on a bearing of $044.5°$.

Worked Example 2

The diagram shows 3 forces. Express each force in component form and find the resultant force.



Solution

Using components that are parallel and perpendicular to the dashed line gives:

$$\text{8 N force} = \begin{pmatrix} 8\cos 30° \\ 8\sin 30° \end{pmatrix}$$

$$\text{6 N force} = \begin{pmatrix} -6\cos 70° \\ -6\sin 70° \end{pmatrix}$$

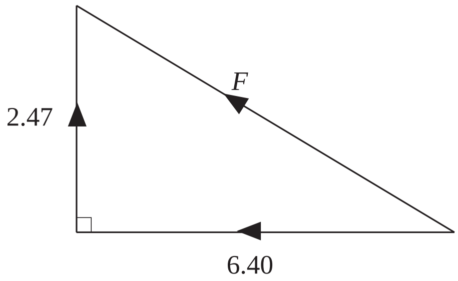
$$\text{12 N force} = \begin{pmatrix} -12\cos 20° \\ 12\sin 20° \end{pmatrix}$$

$$\text{Resultant force} = \begin{pmatrix} 8\cos 30° \\ 8\sin 30° \end{pmatrix} + \begin{pmatrix} -6\cos 70° \\ -6\sin 70° \end{pmatrix} + \begin{pmatrix} -12\cos 20° \\ 12\sin 20° \end{pmatrix}$$

$$= \begin{pmatrix} 8\cos 30° - 6\cos 70° - 12\cos 20° \\ 8\sin 30° - 6\sin 70° + 12\sin 20° \end{pmatrix}$$

$$= \begin{pmatrix} -6.40 \\ 2.47 \end{pmatrix}$$

The diagram below shows the resultant force.



Using Pythagoras,

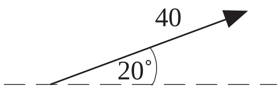
$$F^2 = 2.47^2 + 6.40^2$$

$$F \approx 6.9 \text{ N}$$

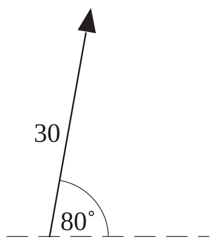
Exercises

1. Write each vector shown below in component form using components parallel and perpendicular to the dashed line.

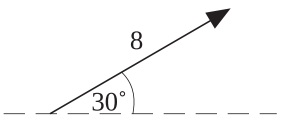
(a)



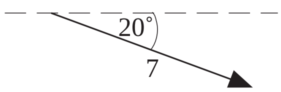
(b)



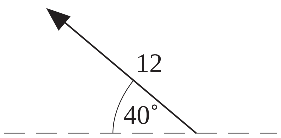
(c)



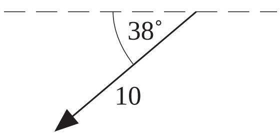
(d)



(e)



(f)



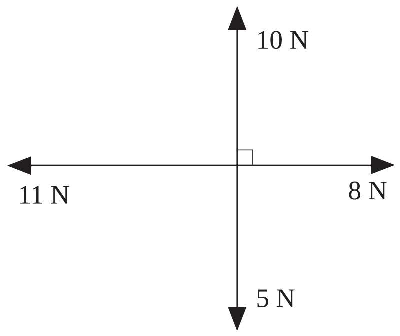
2. For each column vector below, find its magnitude and draw a diagram to show its direction, labelling the size of appropriate angles.

(a) $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ (b) $\begin{pmatrix} -4 \\ 1 \end{pmatrix}$ (c) $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$

(d) $\begin{pmatrix} -6 \\ -6 \end{pmatrix}$ (e) $\begin{pmatrix} 5 \\ -2 \end{pmatrix}$ (f) $\begin{pmatrix} -4 \\ 3 \end{pmatrix}$

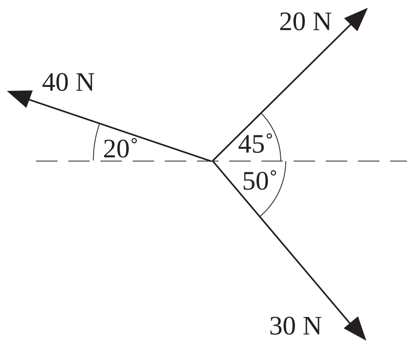
3. The diagram shows 4 forces that act at a point.

Express each force as a column vector, find the resultant force as a column vector and calculate its magnitude.



4. The three forces shown in the diagram act at a point.

Find the magnitude of their resultant and draw a diagram to show its directions.



5. A ship travels at a speed of 3 m/s due east. A current moves the water due south at 1.2 m/s.

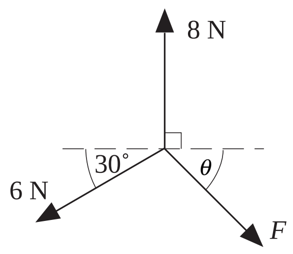
Express the velocity of the ship and the water in column vector form. Find the actual velocity of the ship giving the direction as a bearing.

6. An aeroplane heads on a bearing of $240°$ at a speed of 250 m/s. A wind blows SE at 80 m/s. Find the resultant velocity of the aeroplane, giving the direction as a bearing.

7. An aeroplane wants to fly due south at a speed of 180 m/s. The air in which it will fly is moving NW at 60 m/s. Find the speed at which the aeroplane should fly and the bearing on which it should fly.

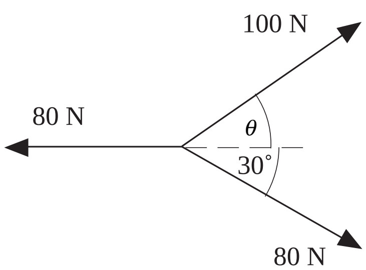
8. The three forces shown in the diagram are in equilibrium, so that their resultant is zero.

Find F and the angle θ.



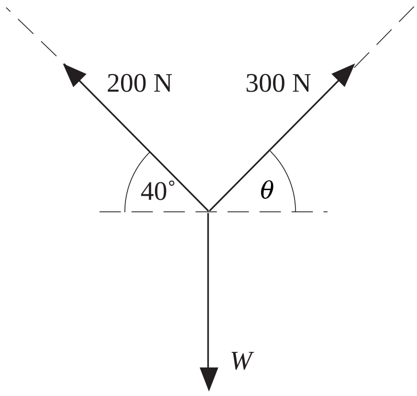
9. The diagram shows three forces.

Find the angle θ if the resultant force acts along the dotted line. Also find the magnitude of the resultant force.

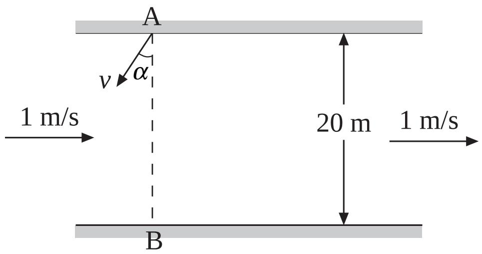


10. The diagram shows the force of gravity, W, that pulls straight down on an object. Two forces of 200 N and 300 N are exerted by cables as shown.

If the resultant of the three forces is zero, find θ and W.



11.



A river is flowing steadily at a speed of 1 m/s. A motor boat leaves point A on one side of the river. The boat has a constant speed, $v = 2$ m/s.

(a) What is the resultant speed of the boat in the direction:

(i) AB;

(ii) downstream?

(b) The boatman wishes to travel to a point B on the other side of the river, directly opposite to A. Calculate the angle α (as shown on the diagram) at which the boat should be steered.

(SEG)

19.5 Commutative and Associative Properties

An operation '*' is said to be *commutative* if $a * b = b * a$. Addition and multiplication of numbers are examples of operations that are commutative, while subtraction and division of numbers are not commutative. For example, if $a = 8$, $b = 5$

$$8 + 5 = 13 = 5 + 8$$
$$8 \times 5 = 40 = 5 \times 8$$

but

$$8 - 5 = 3 \neq 5 - 8 = -3$$
$$8 \div 5 = 1.6 \neq 5 \div 8 = 0.625$$

An operation '*' is *associative* if $a * (b * c) = (a * b) * c$. For the real numbers, addition and multiplication are associative but division and subtraction are not.

Worked Example 1

Show that vector addition is commutative.

Solution

To show that vector addition is commutative, consider the vectors:

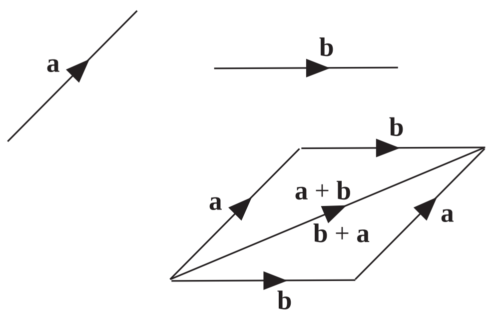
$$\mathbf{a} = \begin{pmatrix} p \\ q \end{pmatrix} \text{ and } \mathbf{b} = \begin{pmatrix} r \\ s \end{pmatrix}$$

Then,

$$\begin{aligned} \mathbf{a} + \mathbf{b} &= \begin{pmatrix} p \\ q \end{pmatrix} + \begin{pmatrix} r \\ s \end{pmatrix} \\ &= \begin{pmatrix} p + r \\ q + s \end{pmatrix} \\ &= \begin{pmatrix} r + p \\ s + q \end{pmatrix} \\ &= \begin{pmatrix} r \\ s \end{pmatrix} + \begin{pmatrix} p \\ q \end{pmatrix} \\ &= \mathbf{b} + \mathbf{a} \end{aligned}$$

As $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$, vector addition is commutative.

Alternatively, a geometric approach is possible, with the vectors **a** and **b** shown below.



From the diagram, $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$, so vector addition is commutative.

Exercises

1. Use the vectors $\mathbf{a} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ to demonstrate that vector subtraction is not commutative.

2. Draw any two vectors **a** and **b**. Draw diagrams to show $\mathbf{a} - \mathbf{b}$ and $\mathbf{b} - \mathbf{a}$. Use your diagrams to explain why vector subtraction is not commutative.

3. Draw three vectors **a**, **b** and **c**. Draw diagrams to show $\mathbf{a} + (\mathbf{b} + \mathbf{c})$ and $(\mathbf{a} + \mathbf{b}) + \mathbf{c}$. Is vector addition associative? Use your diagrams to justify your answer.

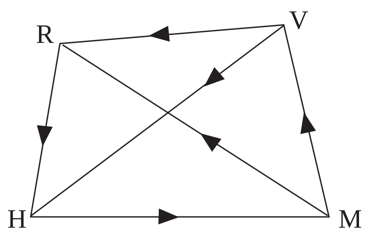
4. Use the vectors $\mathbf{a} = \begin{pmatrix} p \\ q \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} r \\ s \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} u \\ t \end{pmatrix}$ to show that vector addition is associative.

5. Use the vectors $\mathbf{a} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 7 \\ -2 \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} -4 \\ 2 \end{pmatrix}$ to demonstrate that vector addition is not associative.

6. Use a geometrical argument to show that vector subtraction is not associative.

Just for Fun

Jack wants to go to from his home (H) to *three different places, the mountains,* (M), *the village* (V) *and the river* (R) *as shown in the map.*



If he can visit these places in only two given sequences, i.e.

Route A: H $\rightarrow$ M $\rightarrow$ V $\rightarrow$ R $\rightarrow$ H

or

Route B: H $\rightarrow$ M $\rightarrow$ R $\rightarrow$ V $\rightarrow$ H

find out which is the shorter route for Jack to travel.